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REFINED CONTINUUM MODEL OF LAYERED MEDIA WITH THE SLIP ON THE INTERLAYER BOUNDARIES

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Preface

On the basis of an asymptotic method of averaging the new equations of continuum theory of layered media with interlayer slipping are obtained taking in to account the terms of the second order in the small layer thickness parameter. The linear condition of slipping, which connects jumps of tangent shifts on contact borders and tangent tension, is used. The equations received in this study also are asymptotically full generalization of some models of the layered continuum media based on engineering approaches or approximate hypotheses about nature of deformation of layers. Such models are necessary when studying static deformation of a massif and at the solution of dynamic problems of geophysics. Wave properties of the received system of equations are investigated, dispersive ratios for harmonious waves are derived. The propagation of a superficial Rayleigh waves on border of an elastic layered half-space is considered.

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Introduction

Interest in a problem of propagation and transformation of waves in layered continuum media is conjugated with problems of seismology and engineering geophysics. As a rule, seismicity is connected with mountainous areas in which rocky breeds come to a terrestrial surface. Often these breeds contain the regular grids of cracks allowing to consider them as layered structures. Classical researches of wave fields in such environments usually proceed from a continuity of a displacement field. However for enough strong seismic actions it is necessary to consider possibility of slipping and tangent motions on borders of layers. For such extended influences it is necessary to use "average", continual models of continuous media with structure as it is impossible to take into account the deformation of each element of structure.

In our work on the basis of an asymptotic method [1,2] the average equations of the layered continuous media with slipping are derived taking into account members of the second order in the small parameter of thickness of a layer. The linear condition of slipping connecting jumps of tangent shifts on contact borders and tangent tension is used. The equations of zero approach were derived earlier in [3,4]. The new equations received here also can be considered as asymptotically full generalization of models [5-6] of layered media based on engineering approaches or on approximate hypotheses of deformation nature of layers. Such models are necessary when studying static deformation of a massif and at the solution of dynamic (wave) problems of geophysics.

It is also possible to note that the theory of layered environments with slipping on contact borders can be useful at the description of composite materials with additional soft layers (for instance, rubber) between layers from the main, rather rigid elastic material (metal).

Wave properties of the derived system of the equations are investigated, dispersive ratios are received. The solution of a task on a superficial Rayleigh wave on border of an elastic layered half-space is constructed.

1. Derivation of refined theory

In the Cartesian rectangular system of coordinates x_1, x_2, x_3 consider the boundless layered medium. Let axis x_3 is perpendicular to plane-parallel interlayer boundaries.

The interlayer boundaries have coordinates $x_3 = x^{(s)} = s\varepsilon$, $s=0, \pm 1, \pm 2 \ldots$, where constant layer thickness $\varepsilon <<1$ is a small parameter. More exactly, the ratio $\varepsilon / l << 1$ should be small, where l is a characteristic size of distributed loads, for instance, characteristic wave length in dynamic process under consideration. Then all space values have to be made dimensionless using this scale l.

At interlayer boundaries the following slipping conditions should take place according to supposition that interlayer boundary is always compressed:

$\sigma_{33} < 0 \quad [u_3] = [\sigma_{\gamma 3}] = [\sigma_{33}] = 0$

$[\sigma_{\gamma 3}] = k_*[u_\gamma]$ - linear slipping of Winkler type,

$k_*\varepsilon = k = O(1)$

Square brackets $[f] = f\big|_{x^{(s)}+0} - f\big|_{x^{(s)}-0}$ denote the jump of value f at interlayer boundary.

According to Introduction, such conditions are approximately fulfilled, if between layers there present the soft sublayers of thickness δ, $\delta / \varepsilon << 1$, with small shear modulus μ_δ. Obviously in this case:

$$[\sigma_{\gamma 3}] = k[u_\gamma]/\varepsilon = \frac{k\delta}{\varepsilon}\frac{[u_\gamma]}{\delta} = \mu_\delta \frac{[u_\gamma]}{\delta}$$

where $[u_\gamma]/\delta$ is shear deformation of soft sublayer. In this case $\mu_\delta = k\delta/\varepsilon$, or, vise versa, $k = \mu_\delta \varepsilon / \delta$. It is possible to say, that k is a coefficient of shear interlayer connection. The layers themselves are isotropic elastic and are subjected to Hooke's law:

$x_3 \neq x^{(s)}$

$\sigma_{ij,j} - \rho u_{i,tt} = 0$

$\sigma_{ij} = C_{ijkl} u_{k,l}$

Expression for a tensor of modules of elasticity is:

$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

Enter [1] "fast" variable according to method of asymptotic averaging $\xi = x_3 / \varepsilon$. Following [1], consider that $u_k = u_k(x_l, \xi, t)$ is a smooth function regarding «slow» variables x_l and «fast» variable ξ, excluding points $\xi^{(s)} = x^{(s)} / \varepsilon$, where it may have jumps of first kind. Besides, regarding ξ it is 1-periodic: $[[u_i]] = u_i\big|_{\xi^{(s)}+1/2} - u_i\big|_{\xi^{(s)}-1/2} = 0$. Taking into account such choice of arguments and the chain differentiation rules, rewrite the system of equations on frequency cell

$x^{(s)} - 1/2 \le x_3 \le x^{(s)} + 1/2$, $-1/2 \le \xi \le 1/2$:

Equations for $x_3 \ne x^{(s)}$, $\xi \ne 0$:

$$\varepsilon^{-2} C_{i3k3} u_{k,\xi\xi} + \varepsilon^{-1}(C_{ijk3} u_{k,j\xi} + C_{i3kl} u_{k,l\xi}) + C_{ijkl} u_{k,lj} - \rho u_{i,tt} = 0$$

Contact conditions for $x_3 = x^{(s)}$, $\xi = 0$:

$$\varepsilon^{-1} C_{33k3} u_{k,\xi} + C_{33kl} u_{k,l} < 0$$

$$[u_3] = 0, \qquad [\varepsilon^{-1} C_{i3k3} u_{k,\xi} + C_{i3kl} u_{k,l}] = 0$$

$$\varepsilon^{-1} C_{\gamma 3k3} u_{k,\xi} + C_{\gamma 3kl} u_{k,l} = k_*[u_\gamma]$$

1-periodic conditions:

$$[[u_i]] = u_i\big|_{\xi+1/2} - u_i\big|_{\xi-1/2} = 0$$

Here and farther Greek indexes (β, γ) take values 1 and 2, Latin indexes take values 1, 2, 3. Present displacements of the medium in the form of an asymptotic row on degrees of small parameter ε:

$$u_i = u_i^{(0)}(x_k, \xi, t) + \varepsilon u_i^{(1)}(x_k, \xi, t) + \varepsilon^2 u_i^{(2)}(x_k, \xi, t) + \varepsilon^3 u_i^{(3)}(x_k, \xi, t) + \ldots$$

Introduce the operation of «averaging» $\langle f \rangle$ for function of «fast» variable ξ, which will be often used later: $\langle f \rangle = \int_{-1/2}^{1/2} f d\xi$. Displacement approximations should satisfy additional condition $\langle u_k^{(n)} \rangle = 0$ [1].

Substitute this representation in to system of equations of the theory of elasticity. Equating to zero the member at negative degree ε^{-2} get, that zero approach $u_i^{(0)}$ does not depend on «fast» variable ξ: $u_i^{(0)} = w_i(x_k, t)$. Equating to zero the member at negative degree ε^{-1} get, that first approach $u_i^{(1)}$ satisfies to equation: $C_{i3k3}u_{k,\xi\xi}^{(1)} = 0$. The system of differential equations after that takes view:

$$C_{ijkl}w_{k,jl} + C_{ijk3}u_{k,j\xi}^{(1)} + (C_{i3kl}u_{k,l}^{(1)} + C_{i3k3}u_{k,\xi}^{(2)})_{,\xi} +$$
$$+\varepsilon\left[C_{ijkl}u_{k,jl}^{(1)} + C_{ijk3}u_{k,j\xi}^{(2)} + (C_{i3kl}u_{k,l}^{(2)} + C_{i3k3}u_{k,\xi}^{(3)})_{,\xi}\right] +$$
$$+\varepsilon^2\left[C_{ijkl}u_{k,jl}^{(2)} + C_{ijk3}u_{k,j\xi}^{(3)} + (C_{i3kl}u_{k,l}^{(3)} + C_{i3k3}u_{k,\xi}^{(4)})_{,\xi}\right] + \ldots = \rho w_{i,tt} + \varepsilon\rho u_{i,tt}^{(1)} + \varepsilon^2\rho u_{i,tt}^{(2)} + \ldots$$

According to representation for displacements one can get the following representation for stress tensor components:

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \varepsilon\sigma_{ij}^{(1)} + \varepsilon^2\sigma_{ij}^{(2)} + \ldots$$

где $\sigma_{ij}^{(\mathrm{n})} = C_{ijkl}u_{k,l}^{(\mathrm{n})} + C_{ijk3}u_{k,\xi}^{(\mathrm{n+1})}$.

All stress presentations are 1-periodic functions of ξ. In particular, $\sigma_{i3}^{(\mathrm{n})} = C_{i3kl}u_{k,l}^{(\mathrm{n})} + C_{i3k3}u_{k,\xi}^{(\mathrm{n+1})}$ and the following conditions take place $[\sigma_{i3}^{(\mathrm{n})}] = 0$, $[[\sigma_{i3}^{(\mathrm{n})}]] = 0$. Easy to see, that $\left\langle \sigma_{i3}^{(\mathrm{n})}{}_{,\xi} \right\rangle = 0$.

Leaving the system of equations for the members of a certain order ε, applying the averaging operation $\langle f \rangle$ and thus getting rid of "fast" variable, we obtain the averaged effective model of the layered medium with slip of Winkler type.

Now derive a revised theory of second order. For this in the system of equations we keep terms of order ε^2. Applying the operation of averaging on the periodicity cell $\langle\ \rangle$ to the system of equations, get the following result:

$$C_{ijkl}w_{k,jl} + C_{ijk3}\left\langle u_{k,\xi}^{(1)} \right\rangle_{,j} + \varepsilon C_{ijk3}\left\langle u_{k,\xi}^{(2)} \right\rangle_{,j} + \varepsilon^2 C_{ijk3}\left\langle u_{k,\xi}^{(3)} \right\rangle_{,j} = \rho w_{i,tt}$$

This is the desired averaged system of equations for a layered medium with slippage, for complete formulation it needs to find functions $\left\langle u_{k,\xi}^{(\mathrm{n})} \right\rangle$ *(n=1,2,3),* which participate in this system.

Each of functions $u_i^{(\mathrm{n})}(x_k, \xi, t)$ *(n=1,2,3),* may be found from appropriate «task on the periodicity cell» at $-1/2 \le \xi \le 1/2$ [1], which arised from equating to zero the terms of definite degree ε^{n-1} in asymptotic system of equations. Additional conditions for these functions detecting are obtained by contact condition reformulation for each from these functions on interlayer boundaries, 1-periodic conditions $[[u_i^{(n)}]]=0$ and conditions $\left\langle u_i^{(n)}\right\rangle=0$.

Let's formulate these three tasks on the cell $-1/2\le\xi\le 1/2$.

Task on the cell 1

For $|\xi|<1/2$ have

$$C_{i3k3}u^{(1)}_{k,\xi\xi}=0$$

For $\xi=0$ have

$$[C_{i3k3}u^{(1)}_{k,\xi}]=0,\qquad [u_3^{(1)}]=0$$

$$k[u_\gamma^{(1)}]=C_{\gamma 3kl}w_{k,l}+C_{\gamma 3k3}u^{(1)}_{k,\xi}$$

Additional conditions:

$$[[u_i^{(1)}]]=0,\ \left\langle u_i^{(1)}\right\rangle=0$$

Solution of task 1

а) $i=\gamma$

$$C_{\gamma 3k3}=\mu\delta_{\gamma k}$$

$u^{(1)}_{\gamma,\xi\xi}=0$ при $|\xi|<1/2$

For $\xi=0$

$$[u^{(1)}_{\gamma,\xi}]=0,\quad k[u_\gamma^{(1)}]=\mu(w_{\gamma,3}+w_{3,\gamma})+\mu u^{(1)}_{\gamma,\xi},\ [[u_\gamma^{(1)}]]=0,\ \left\langle u_\gamma^{(1)}\right\rangle=0$$

Function $u_\gamma^{(1)}=\varphi_\gamma\xi+c_\gamma{}^{\pm}$ is the solution of differential equation. Then: from condition $[[u_\gamma^{(1)}]]=0$ follows $\varphi_\gamma/2+c_\gamma{}^{+}=-\varphi_\gamma/2+c_\gamma{}^{-}$. From this follows, that $[u_\gamma^{(1)}]=c_\gamma{}^{+}-c_\gamma{}^{-}=-\varphi_\gamma$. From condition $\left\langle u_\gamma^{(1)}\right\rangle=0$ follows $c_\gamma{}^{+}+c_\gamma{}^{-}=0$, or $c_\gamma{}^{\pm}=\mp\varphi_\gamma/2$, or $u_\gamma^{(1)}=\varphi_\gamma(\xi\mp 1/2)$.

Condition for jump of shear displacements takes the view: $-k\varphi_\gamma=\mu(w_{\gamma,3}+w_{3,\gamma})+\mu\varphi_\gamma$, from this follows, that $\varphi_\gamma=-\tau_\gamma/(k+\mu)$, $\tau_\gamma=\mu(w_{\gamma,3}+w_{3,\gamma})$.

б) $i=3$

$C_{33k3} = (\lambda + 2\mu)\delta_{3k}$

$u^{(1)}_{3,\xi\xi} = 0$ for $|\xi| < 1/2$

For $\xi = 0$

$[u^{(1)}_{3,\xi}] = 0$, $[u^{(1)}_3] = 0$, $[[u^{(1)}_3]] = 0$, $\langle u^{(1)}_3 \rangle = 0$

Solution of this task is trivial: $u^{(1)}_3 = 0$.

So, the solution of task 1 on periodicity cell is represented by functions:

$u^{(1)}_\gamma = \varphi_\gamma(\xi \mp 1/2)$

$u^{(1)}_3 = 0$

Here and farther in formulas the upper sign in symbol $\mp$ relates to values $0 < \xi < 1/2$, and lower sign relates to values $-1/2 < \xi < 0$. Also $\varphi_\gamma = -\tau_\gamma / (k + \mu)$, $\tau_\gamma = \mu(w_{\gamma,3} + w_{3,\gamma})$. Accordingly, the derivatives of the averaged output required for the system have the expressions

$u^{(1)}_{3,\xi} = 0$, $u^{(1)}_{\gamma,\xi} = \varphi_\gamma$, $\langle u^{(1)}_{3,\xi} \rangle = 0$, $\langle u^{(1)}_{\gamma,\xi} \rangle = \varphi_\gamma$

<u>*Task on cell 2*</u>

For $|\xi| < 1/2$:

$C_{ijkl} w_{k,jl} + C_{ijk3} u^{(1)}_{k,j\xi} + (C_{i3kl} u^{(1)}_{k,l} + C_{i3k3} u^{(2)}_{k,\xi})_{,\xi} = \rho w_{i,tt}$

Applying to this differential equation the averaging operation $\langle\ \rangle$ and taking into account, that $\langle (C_{i3kl} u^{(1)}_{k,l} + C_{i3k3} u^{(2)}_{k,\xi})_{,\xi} \rangle = 0$, and other members of this equation do not depend on ξ, get its more simple consequence:

$C_{i3k3} u^{(2)}_{k,\xi\xi} = -C_{i3kl} u^{(1)}_{k,\xi l}$

For $\xi = 0$:

$[C_{i3k3} u^{(2)}_{k,\xi}] = -[C_{i3kl} u^{(1)}_{k,l}]$, $\quad [u^{(2)}_3] = 0$

$k[u^{(2)}_\gamma] = C_{\gamma 3kl} u^{(1)}_{k,l} + C_{\gamma 3k3} u^{(2)}_{k,\xi}$

Additional conditions:

$[[u_i^{(2)}]] = 0$, $\left\langle u_i^{(2)} \right\rangle = 0$

<u>*Solution of task 2*</u>

a) $i = \gamma$

$u_{\gamma,\xi\xi}^{(2)} = -\psi_\gamma$, $\psi_\gamma = \varphi_{\gamma,3}$ при $|\xi| < 1/2$

For $\xi = 0$

$[u_{\gamma,\xi}^{(2)}] = -[u_{\gamma,3}^{(1)}] = \psi_\gamma$, $k[u_\gamma^{(2)}] = \mu(u_{\gamma,3}^{(1)} + u_{\gamma,\xi}^{(2)}) = \mu(-\psi_\gamma / 2 + b_\gamma^+)$, $[[u_\gamma^{(2)}]] = 0$, $\left\langle u_\gamma^{(2)} \right\rangle = 0$

Solution of the differential equation is a function $u_\gamma^{(2)} = -\psi_\gamma \xi^2 / 2 + b_\gamma^\pm \xi + c_\gamma{}^\pm$.

From condition $[[u_\gamma^{(2)}]] = 0$ follows $-\psi_\gamma / 8 + b_\gamma^+ / 2 + c_\gamma{}^+ = -\psi_\gamma / 8 - b_\gamma^- / 2 + c_\gamma{}^-$.

From this follows, that $[u_\gamma^{(2)}] = c_\gamma{}^+ - c_\gamma{}^- = -(b_\gamma^+ + b_\gamma^-) / 2$.

From condition $[u_{\gamma,\xi}^{(2)}] = -[u_{\gamma,3}^{(1)}] = \psi_\gamma$ follows $b_\gamma^+ - b_\gamma^- = \psi_\gamma$.

Then, $-k(b_\gamma^+ + b_\gamma^-) / 2 = \mu(-\psi_\gamma / 2 + b_\gamma^+)$, and, as consequence

$-k(2b_\gamma^+ - \psi_\gamma) / 2 = \mu(-\psi_\gamma / 2 + b_\gamma^+)$,

$b_\gamma^\pm = \pm\psi_\gamma / 2$, $[u_\gamma^{(2)}] = 0$, $c_\gamma{}^+ = c_\gamma{}^- = c_\gamma$.

New representation of the solution takes view: $u_\gamma^{(2)} = -\psi_\gamma(\xi^2 \mp \xi + c_\gamma) / 2$.

Condition $\left\langle u_\gamma^{(2)} \right\rangle = 0$ gives: $\xi^3 / 3\Big|_{-1/2}^{1/2} + \xi^2 / 2\Big|_{-1/2}^{0} - \xi^2 / 2\Big|_{0}^{1/2} + c_\gamma\, \xi\Big|_{-1/2}^{1/2} = 0$ or $c_\gamma = 1/6$.

Finally $u_\gamma^{(2)} = -\psi_\gamma(\xi^2 \mp \xi + 1/6) / 2$.

б) $i = 3$

$u_{3,\xi\xi}^{(2)} = -\lambda u_{\beta,\xi\beta}^{(1)} / (\lambda + 2\mu) = -\psi_3$, $\psi_3 = \lambda\varphi_{\beta,\beta} / (\lambda + 2\mu)$ при $|\xi| < 1/2$

For $\xi = 0$

$[u_{3,\xi}^{(2)}] = -[\lambda u_{\beta,\beta}^{(1)} / (\lambda + 2\mu)] = \psi_3$, $[u_3^{(2)}] = 0$, $[[u_3^{(2)}]] = 0$, $\left\langle u_3^{(2)} \right\rangle = 0$

The solution of this task is analogical to considered above and has the view:

$u_3^{(2)} = -\psi_3(\xi^2 \mp \xi + 1/6) / 2$

So, final solution of this task on periodicity cell is represented by following functions:

$u_\gamma^{(2)} = -\psi_\gamma(\xi^2 \mp \xi + 1/6)/2$

$u_3^{(2)} = -\psi_3(\xi^2 \mp \xi + 1/6)/2$

where $\psi_\gamma = \varphi_{\gamma,3}$, $\psi_3 = \lambda\varphi_{\beta,\beta}/(\lambda+2\mu)$

Derivatives, required for averaged system of equation derivation, are

$u_{\gamma,\xi}^{(2)} = -\psi_\gamma(\xi \mp 1/2)$, $u_{3,\xi}^{(2)} = -\psi_3(\xi \mp 1/2)$, $\langle u_{3,\xi}^{(2)} \rangle = 0$, $\langle u_{\gamma,\xi}^{(2)} \rangle = 0$

From this follows, that terms of second approach of displacements does not participate in averaged system of equations..

__Tak on cell 3__

For $|\xi| < 1/2$

$C_{i3k3}u_{k,\xi\xi}^{(3)} = -C_{ijkl}u_{k,jl}^{(1)} - C_{i3kl}u_{k,\xi l}^{(2)} - C_{ijk3}u_{k,\xi j}^{(2)} + \rho u_{i,tt}^{(1)}$

For $\xi = 0$

$[C_{i3k3}u_{k,\xi}^{(3)}] = -[C_{i3kl}u_{k,l}^{(2)}]$, $[u_3^{(3)}] = 0$

$k[u_\gamma^{(3)}] = C_{\gamma 3kl}u_{k,l}^{(2)} + C_{\gamma 3k3}u_{k,\xi}^{(3)}$

Additional conditions:

$[[u_i^{(3)}]] = 0$, $\langle u_i^{(3)} \rangle = 0$

__Solution of task 3__

a) $i = \gamma$

Expand the expression for the tensor of elastic moduli:

$$C_{ijkl}u_{k,jl}^{(1)} = C_{\gamma j\beta l}u_{\beta,jl}^{(1)} = (\lambda\delta_{\gamma j}\delta_{\beta l} + \mu\delta_{\gamma\beta}\delta_{jl} + \mu\delta_{\gamma l}\delta_{j\beta})u_{\beta,jl}^{(1)} = (\lambda+\mu)u_{\beta,\beta\gamma}^{(1)} + \mu u_{\gamma,ll}^{(1)}$$

$$(C_{\gamma 3kl} + C_{\gamma lk3})u_{k,\xi l}^{(2)} = \left((\lambda+\mu)\delta_{\gamma l}\delta_{3k} + 2\mu\delta_{\gamma k}\delta_{3l}\right)u_{k,\xi l}^{(2)} = (\lambda+\mu)u_{3,\xi\gamma}^{(2)} + 2\mu u_{\gamma,\xi 3}^{(2)}$$

Equation for task on cell:

$u_{\gamma,\xi\xi}^{(3)} = u_{\gamma,ll}^{(1)} - (\lambda+\mu)u_{\beta,\beta\gamma}^{(1)}/\mu - 2u_{\gamma,\xi 3}^{(2)} - (\lambda+\mu)u_{3,\xi\gamma}^{(2)}/\mu + \rho u_{i,tt}^{(1)}/\mu$ при $|\xi| < 1/2$

For $\xi = 0$

$[u_{\gamma,\xi}^{(3)}] = -[u_{\gamma,3}^{(2)} + u_{3,\gamma}^{(2)}] = 0$, $k[u_\gamma^{(3)}] = \mu(u_{\gamma,3}^{(2)} + u_{3,\gamma}^{(2)} + u_{\gamma,\xi}^{(3)})$, $[[u_\gamma^{(3)}]] = 0$, $\langle u_\gamma^{(3)} \rangle = 0$

Equation takes view:

$$u^{(3)}_{\gamma,\xi\xi} = \chi_\gamma\left(\xi \mp 1/2\right)$$

where $\chi_\gamma = -\varphi_{\gamma,ll} - (\lambda+\mu)\varphi_{\beta,\beta\gamma}/\mu + 2\psi_{\gamma,3} + (\lambda+\mu)\psi_{3,\gamma}/\mu + \rho\varphi_{\gamma,tt}/\mu$

Integrating and taking in to account condition $[u^{(3)}_{\gamma,\xi}] = 0$, get

$$u^{(3)}_\gamma = \chi_\gamma\left(\xi^3/6 \mp \xi^2/4 + b_\gamma\xi + c^\pm_\gamma\right)$$

From this follows:

$$[u^{(3)}_\gamma] = \chi_\gamma\left(c^+_\gamma - c^-_\gamma\right) = \chi_\gamma\left(1/12 - b_\gamma\right).$$

Then:

$$k[u^{(3)}_\gamma] = k\chi_\gamma\left(1/12 - b_\gamma\right) = \mu(\chi_\gamma b_\gamma - \psi_{\gamma,3}/12 - \psi_{3,\gamma}/12)$$

$$\chi_\gamma b_\gamma = \left(k\chi_\gamma + \mu\psi_{\gamma,3} + \mu\psi_{3,\gamma}\right)/(k+\mu)/12$$

Desired equation takes form:

$$u^{(3)}_{\gamma,\xi} = \chi_\gamma\left(\xi^2 \mp \xi\right)/2 + \left(k\chi_\gamma + \mu\psi_{\gamma,3} + \mu\psi_{3,\gamma}\right)/(k+\mu)/12$$

For averaged derivative get:

$$\left\langle u^{(3)}_{\gamma,\xi}\right\rangle = \mu\left(\psi_{\gamma,3} + \psi_{3,\gamma} - \chi_\gamma\right)/(k+\mu)/12$$

Finally, after obvious transformations have:

$$\left\langle u^{(3)}_{\gamma,\xi}\right\rangle = \mu\left(\varphi_{\gamma,\beta\beta} + (3\lambda+2\mu)\varphi_{\beta,\beta\gamma}/(\lambda+2\mu) - \rho\varphi_{\gamma,tt}/\mu\right)/(k+\mu)/12$$

б) $i = 3$

Expand the expression for the tensor of elastic moduli:

$$C_{3jkl}u^{(1)}_{k,jl} = C_{3j\beta l}u^{(1)}_{\beta,jl} = (\lambda\delta_{3j}\delta_{\beta l} + \mu\delta_{3\beta}\delta_{jl} + \mu\delta_{3l}\delta_{j\beta})u^{(1)}_{\beta,jl} = (\lambda+\mu)u^{(1)}_{\beta,\beta 3}$$

$$(C_{33kl} + C_{3lk3})u^{(2)}_{k,\xi l} = \left((\lambda+3\mu)\delta_{3l}\delta_{3k} + (\lambda+\mu)\delta_{kl}\right)u^{(2)}_{k,\xi l} = 2(\lambda+2\mu)u^{(2)}_{3,\xi 3} + (\lambda+\mu)u^{(2)}_{\beta,\xi\beta}$$

Equation on cell:

$$u^{(3)}_{3,\xi\xi} = -(\lambda+\mu)u^{(1)}_{\beta,\beta 3}/(\lambda+2\mu) - 2u^{(2)}_{3,\xi 3} - (\lambda+\mu)u^{(2)}_{\beta,\xi\beta}/(\lambda+2\mu) \text{ при } \left|\xi\right| < 1/2$$

При $\xi = 0$

$$[u^{(3)}_{3,\xi}] = -[u^{(2)}_{3,3}] - \lambda[u^{(2)}_{\beta,\beta}]/(\lambda+2\mu) = 0,\ [u^{(3)}_3] = 0,\ [[u^{(3)}_3]] = 0,\ \left\langle u^{(3)}_3\right\rangle = 0$$

Equation may be rewritten as:

$u^{(3)}_{3,\xi\xi} = \chi_3\left(\xi \mp 1/2\right)$

where $\chi_3 = (\lambda+\mu)\psi_{\beta,\beta}/(\lambda+2\mu) + 2\psi_{3,3} - (\lambda+\mu)\varphi_{\beta,\beta 3}/(\lambda+2\mu)$

Using $[u^{(3)}_3] = 0$ и $[u^{(3)}_{3,\xi}] = 0$ get:

$u^{(3)}_3 = \chi_3\left(\xi^3/6 \mp \xi^2/4 + b_3\xi + c_3\right)$

From condition $[[u^{(3)}_3]] = 0$ follows:

$-1/48 + 1/16 - b_3/2 + c_3 = 1/48 - 1/16 + b_3/2 + c_3$ или $b_3 = 1/12$

From this get:

$u^{(3)}_{3,\xi} = \chi_3\left(\xi^2/2 \mp \xi/2 + 1/12\right)$

Easy to see, that

$$\left\langle u^{(3)}_{3,\xi}\right\rangle = \chi_3\left(\xi^3/6 \mp \xi^2/4 + \xi/12\right)\Big|_{-1/2}^{1/2} = \chi_3(1/24 + 1/12 - 1/16 - 1/16) = 0$$

So, the desired solution on periodicity cell is represented by following functions (let's write only required derivatives by "fast" variable):

$u^{(3)}_{\gamma,\xi} = \chi_\gamma(\xi^2 \mp \xi)/2 + (\mu\psi_{\gamma,3} + \mu\psi_{3,\gamma} + k\chi_\gamma)/(k+\mu)/12$

$u^{(3)}_{3,\xi} = \chi_3(\xi^2 \mp \xi + 1/6)/2$,

where

$$\chi_\gamma = 2\psi_{\gamma,3} + (\lambda+\mu)\psi_{3,\gamma}/\mu - \varphi_{\gamma,kk} - (\lambda+\mu)\varphi_{\beta,\beta\gamma}/\mu + \rho\varphi_{\gamma,tt}/\mu$$
$$\chi_3 = (\lambda+\mu)\psi_{\beta,\beta}/(\lambda+2\mu) + 2\psi_{3,3} - (\lambda+\mu)\varphi_{\beta,\beta 3}/(\lambda+2\mu)$$

Expressions for averaged derivatives have the view:

$$\left\langle u^{(3)}_{\gamma,\xi}\right\rangle = \frac{1}{12}\frac{\mu}{(k+\mu)}\left(\psi_{\gamma,3} + \psi_{3,\gamma} - \chi_\gamma\right), \qquad \left\langle u^{(3)}_{3,\xi}\right\rangle = 0$$

Final expression $\left\langle u^{(3)}_{\gamma,\xi}\right\rangle$ may be written as:

$$\left\langle u^{(3)}_{\gamma,\xi}\right\rangle = \frac{1}{12}\frac{\mu}{(k+\mu)}\left(\varphi_{\gamma,\beta\beta} + \frac{3\lambda+2\mu}{\lambda+2\mu}\varphi_{\beta,\beta\gamma} - \frac{\rho}{\mu}\varphi_{\gamma,tt}\right)$$

2. Various variants of averaged system of equations

Using these results we formulate the desired system of equations (Latin indexes *i*,*j*,*k*,*l* =1,2,3, Greek indexes β, γ=1,2):

$$C_{\gamma jkl} w_{k,jl} + C_{\gamma jk3} \left\langle u^{(1)}_{k,\xi} \right\rangle_{,j} + \varepsilon^2 C_{\gamma jk3} \left\langle u^{(3)}_{k,\xi} \right\rangle_{,j} = \rho w_{\gamma,tt}$$

$$C_{3jkl} w_{k,jl} + C_{3jk3} \left\langle u^{(1)}_{k,\xi} \right\rangle_{,j} + \varepsilon^2 C_{3jk3} \left\langle u^{(3)}_{k,\xi} \right\rangle_{,j} = \rho w_{3,tt}$$

Taking into account the expression for the tensor of elastic moduli, the terms of this system may be written in the form:

$$C_{\gamma jkl} w_{k,jl} = (\lambda+\mu) w_{k,k\gamma} + \mu w_{\gamma,kk}, \qquad C_{3jkl} w_{k,jl} = (\lambda+\mu) w_{k,k3} + \mu w_{3,kk}$$

$$C_{\gamma jk3} \left\langle u^{(1)}_{k,\xi} \right\rangle_{,j} = C_{\gamma j\beta 3} \left\langle u^{(1)}_{\beta,\xi} \right\rangle_{,j} = \mu \varphi_{\gamma,3}, \qquad C_{3jk3} \left\langle u^{(1)}_{k,\xi} \right\rangle_{,j} = C_{3j\beta 3} \left\langle u^{(1)}_{\beta,\xi} \right\rangle_{,j} = \mu \varphi_{\beta,\beta}$$

$$C_{\gamma jk3} \left\langle u^{(3)}_{k,\xi} \right\rangle_{,j} = \mu \left\langle u^{(3)}_{\gamma,\xi} \right\rangle_{,3} = \mu^2 \left(\varphi_{\gamma,\beta\beta 3} + (3\lambda+2\mu) \varphi_{\beta,\beta\gamma 3} / (\lambda+2\mu) - \rho \varphi_{\gamma,tt3} / \mu \right) / (k+\mu) / 12$$

$$C_{3jk3} \left\langle u^{(3)}_{k,\xi} \right\rangle_{,j} = \left\langle u^{(3)}_{\beta,\xi} \right\rangle_{,\beta} = \mu^2 \left(4(\lambda+\mu) \varphi_{\beta,\beta\alpha\alpha} / (\lambda+2\mu) - \rho \varphi_{\beta,\beta tt} / \mu \right) / (k+\mu) / 12$$

Finally refined system of equations looks like this:

$$(\lambda+\mu) w_{k,k\gamma} + \mu w_{\gamma,kk} + \mu \varphi_{\gamma,3} + \varepsilon^2 \mu^2 \left(\varphi_{\gamma,\beta\beta 3} + (3\lambda+2\mu) \varphi_{\beta,\beta\gamma 3} / (\lambda+2\mu) - \rho \varphi_{\gamma,tt3} / \mu \right) / (k+\mu) / 12 = \rho w_{\gamma,tt}$$

$$(\lambda+\mu) w_{k,k3} + \mu w_{3,kk} + \mu \varphi_{\beta,\beta} + \varepsilon^2 \mu^2 \left(4(\lambda+\mu) \varphi_{\beta,\beta\alpha\alpha} / (\lambda+2\mu) - \rho \varphi_{\beta,\beta tt} / \mu \right) / (k+\mu) / 12 = \rho w_{3,tt}$$

Remind, that $\varphi_\gamma = -\mu (w_{\gamma,3} + w_{3,\gamma}) / (k+\mu)$. In general system the expression for φ_γ is not substituted to avoid appearance of very complex formulas. It is clear, that finally we have the system of 4th order for displacement field w_k along spacial coordinates, containing also mixed derivatives in time.

The system of equations become much more simple for case of ideal slipping interlayer contact $k=0$.

$$(\lambda+\mu) w_{k,k\gamma} + \mu w_{\gamma,kk} + \mu \varphi_{\gamma,3} + \varepsilon^2 \mu \left(\varphi_{\gamma,\beta\beta 3} + (3\lambda+2\mu) \varphi_{\beta,\beta\gamma 3} / (\lambda+2\mu) - \rho \varphi_{\gamma,tt3} / \mu \right) / 12 = \rho w_{\gamma,tt}$$

$$(\lambda+\mu) w_{k,k3} + \mu w_{3,kk} + \mu \varphi_{\beta,\beta} + \varepsilon^2 \mu \left(4(\lambda+\mu) \varphi_{\beta,\beta\alpha\alpha} / (\lambda+2\mu) - \rho \varphi_{\beta,\beta tt} / \mu \right) / 12 = \rho w_{3,tt}$$

$$\varphi_\gamma = -(w_{\gamma,3} + w_{3,\gamma})$$

Quasi-static case for general system is obtained by removal terms with time derivatives:

$$(\lambda+\mu)w_{k,k\gamma}+\mu w_{\gamma,kk}+\mu\varphi_{\gamma,3}+\varepsilon^2\mu^2\left(\varphi_{\gamma,\beta\beta3}+(3\lambda+2\mu)\varphi_{\beta,\beta\gamma3}/(\lambda+2\mu)\right)/(k+\mu)/12=0$$

$$(\lambda+\mu)w_{k,k3}+\mu w_{3,kk}+\mu\varphi_{\beta,\beta}+\varepsilon^2\mu^2(\lambda+\mu)\varphi_{\beta,\beta\alpha\alpha}/(\lambda+2\mu)/(k+\mu)/3=0$$

$$\varphi_\gamma=-\mu(w_{\gamma,3}+w_{3,\gamma})/(k+\mu)$$

Quasi-static case for system with ideal interlayer slipping gives:

$$(\lambda+\mu)w_{k,k\gamma}+\mu w_{\gamma,kk}+\mu\varphi_{\gamma,3}+\varepsilon^2\mu\left(\varphi_{\gamma,\beta\beta3}+(3\lambda+2\mu)\varphi_{\beta,\beta\gamma3}/(\lambda+2\mu)\right)/12=0$$

$$(\lambda+\mu)w_{k,k3}+\mu w_{3,kk}+\mu\varphi_{\beta,\beta}+\varepsilon^2\mu(\lambda+\mu)\varphi_{\beta,\beta\alpha\alpha}/(\lambda+2\mu)/3=0$$

$$\varphi_\gamma=-(w_{\gamma,3}+w_{3,\gamma})$$

Separately let’s formulate flat (2-dimentional) dynamic system of equations:

$$(\lambda+2\mu)w_{1,11}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{3,13}+\frac{k\mu}{k+\mu}w_{1,33}-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1133}+w_{3,3111}\right)+\rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,33tt}+w_{3,31tt}\right)=\rho w_{1,tt}$$

$$(\lambda+2\mu)w_{3,33}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{1,13}+\frac{k\mu}{k+\mu}w_{3,11}-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1113}+w_{3,1111}\right)+\rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,13tt}+w_{3,11tt}\right)=\rho w_{3,tt}$$

Quasi-static 2D system of equations has the following view:

$$(\lambda+2\mu)w_{1,11}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{3,13}+\frac{k\mu}{k+\mu}w_{1,33}-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1133}+w_{3,3111}\right)=0$$

$$(\lambda+2\mu)w_{3,33}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{1,13}+\frac{k\mu}{k+\mu}w_{3,11}-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1113}+w_{3,1111}\right)=0$$

And, finally, 1D dynamic and quasi-static equation for bending of layered massif (case $w_1=0$, $w_3=w_3(x_1,t)$) takes the following view:

$$\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}w_{3,1111}-\frac{k\mu}{k+\mu}w_{3,11}-\rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}w_{3,11tt}+\rho w_{3,tt}=0 \quad \text{(dynamics)}$$

$$\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}w_{3,1111}-\frac{k\mu}{k+\mu}w_{3,11}=0 \quad \text{(quasi-statics)}$$

Formulas for stress tensor components look like this:

$$\sigma_{ij}^{(0)} = C_{ijkl} w_{k,l} + C_{ijk3} u_{k,\xi}^{(1)}, \quad \sigma_{ij}^{(0)} = \lambda \delta_{ij} w_{k,k} + \mu(w_{i,j} + w_{j.i}) + \mu(\varphi_i \delta_{j3} + \varphi_j \delta_{i3})$$

$$\sigma_{ij}^{(1)} = C_{ijkl} u_{k,l}^{(1)} + C_{ijk3} u_{k,\xi}^{(2)}, \quad \sigma_{ij}^{(1)} = \left(\lambda \delta_{ij} \varphi_{k,k} + \mu(\varphi_{i,j} + \varphi_{j.i}) - \lambda \delta_{ij} \psi_3 - \mu(\psi_i \delta_{j3} + \psi_j \delta_{i3})\right)(\xi \mp 1/2)$$

where

$$\varphi_3 = 0, \quad \varphi_\gamma = -\mu(w_{\gamma,3} + w_{3,\gamma})/(k+\mu), \quad \psi_\gamma = \varphi_{\gamma,3}, \quad \psi_3 = \lambda \varphi_{\beta,\beta}/(\lambda + 2\mu)$$

The boundary conditions for the loaded surface can be written as:

$$\sigma_{ij}^{(0)} \cdot n_j = P_i \quad , \quad \sigma_{ij}^{(1)} \cdot n_j = 0$$

The condition of the first order becomes an identity in some problems with certain orientations of the normal to the boundary.. In this case, the second-order condition: $\sigma_{ij}^{(2)} \cdot n_j = 0$ should be used.

3. Wave properties of the layered media with interlayer slipping.

3.1. Flat harmonic waves.

Define the properties of harmonic waves propagating in an arbitrary direction with respect to the orientation of the layers with arbitrary coupling coefficient layers k. 2D dynamic system of equations for layered medium has the view:

$$(\lambda+2\mu)w_{1,11} + \left(\lambda + \frac{k\mu}{k+\mu}\right)w_{3,13} + \frac{k\mu}{k+\mu} w_{1,33} - \frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1133} + w_{3,3111}\right) + \rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,33tt} + w_{3,31tt}\right) = \rho w_{1,tt}$$

$$(\lambda+2\mu)w_{3,33} + \left(\lambda + \frac{k\mu}{k+\mu}\right)w_{1,13} + \frac{k\mu}{k+\mu} w_{3,11} - \frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1113} + w_{3,1111}\right) + \rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,13tt} + w_{3,11tt}\right) = \rho w_{3,tt}$$

These equations may be rewritten as following:

$$(\lambda+2\mu)w_{1,11} + \lambda w_{3,13} + \tilde{\mu}(w_{1,3} + w_{3,1})_{,3} - \varepsilon^2\mu\beta_1\left(w_{1,3} + w_{3,1}\right)_{,113} + \rho\varepsilon^2\beta_2\left(w_{1,3} + w_{3,1}\right)_{,3tt} = \rho w_{1,tt}$$

$$(\lambda+2\mu)w_{3,33} + \lambda w_{1,13} + \tilde{\mu}(w_{1,3} + w_{3,1})_{,1} - \varepsilon^2\mu\beta_1\left(w_{1,3} + w_{3,1}\right)_{,111} + \rho\varepsilon^2\beta_2\left(w_{1,3} + w_{3,1}\right)_{,1tt} = \rho w_{3,tt}$$

Introduce additional variables:

$$U = w_{1,3} + w_{3,1}$$

$V = \tilde{\mu}u - \varepsilon^2 \mu\beta_1 u_{,11} + \rho\varepsilon^2 \beta_2 u_{,tt}$

Then the system of equation take the following form:

$$\left((\lambda+2\mu)w_{1,11} - \rho w_{1,tt}\right) + \lambda w_{3,13} + V_{,3} = 0$$

$$\lambda w_{1,13} + \left((\lambda+2\mu)w_{3,33} - \rho w_{3,tt}\right) + V_{,1} = 0$$

$$w_{1,3} + w_{3,1} - U = 0$$

$$\tilde{\mu}u - \varepsilon^2 \beta_2 (\mu_* u_{,11} + \rho u_{,tt}) - V = 0$$

Here the new coefficients are introduced:

$$\tilde{\mu} = \mu \frac{k}{k+\mu}, \quad \beta = \frac{\mu}{k+\mu}, \quad \beta_1 = \frac{\lambda+\mu}{\lambda+2\mu}\beta^2 / 3, \quad \beta_2 = \beta^2 / 12, \quad \mu_* = \mu\beta_1 / \beta_2$$

Let's seek the solution of the system as harmonic waves propagating along $\mathbf{n} = (n_1, n_3)$ with frequency ω and wave number $\boldsymbol{\kappa} = \kappa\mathbf{n} = (\kappa_1, \kappa_3)$, $\kappa_1 = \kappa n_1$, $\kappa_3 = \kappa n_3$, $|\boldsymbol{\kappa}| = \kappa$, $|\mathbf{n}| = 1$:

$$w_1 = Ae^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)}, \quad w_3 = Be^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)}$$

$$U = Ce^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)}, \quad V = De^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)}$$

The wave number $\kappa = 2\pi / l$, where l - the length of harmonic wave. We also have $\varepsilon\kappa = 2\pi(\varepsilon / l)$, $\varepsilon^2\kappa^2 = 4\pi^2(\varepsilon / l)^2$. The value ε / l <1 should be a small parameter.

In the result we get homogeneous algebraic system of equations:

$$\left((\lambda+2\mu)\kappa_1^2 + \mu_\varepsilon\kappa_3^2 - \rho\omega^2\right)A + (\lambda+\mu_\varepsilon)\kappa_1\kappa_3 B = 0$$

$$(\lambda+\mu_\varepsilon)\kappa_1\kappa_3 A + \left((\lambda+2\mu)\kappa_3^2 + \mu_\varepsilon\kappa_1^2 - \rho\omega^2\right)B = 0$$

where $\mu_\varepsilon = \tilde{\mu} + \varepsilon^2\beta_2(\mu_*\kappa_1^2 - \rho\omega^2)$.

Condition for the solvability of this algebraic system gives an equation for the propagation velocities of harmonic waves in a layered medium under study:

$$\zeta^4 - \left(1 + \frac{\mu_\varepsilon}{(\lambda+2\mu)}\right)\zeta^2 + \frac{\mu_\varepsilon}{(\lambda+2\mu)} + 4\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{(\mu-\mu_\varepsilon)}{(\lambda+2\mu)}n_1^2 n_3^2 = 0$$

where $\zeta^2 = \rho c^2 / (\lambda+2\mu) = c^2 / c_1^2$, $c = \omega / \kappa$ is the phase velocity of wave propagation in layered media, $c_1 = \sqrt{(\lambda+2\mu)/\rho}$ и $c_2 = \sqrt{\mu / \rho}$ are propagation velocities of elastic longitudinal and transverse waves in a homogeneous elastic medium. Define the direction of the wave by the angle α, $n_1 = \sin\alpha$. For some values of biquadratic equation has exact solutions.

For $\alpha = 0$:

Quasi-longitudinal wave $\zeta_1 = 1$, quasi-transverse wave $\zeta_2 = \sqrt{\tilde{\mu}} \Big/ \sqrt{(\lambda + 2\mu)(1+\varepsilon^2\kappa^2\beta_2)}$.

For $\alpha = \pi/4$:

Quasi-longitudinal wave $\zeta_1 = \sqrt{(\lambda + \mu + \tilde{\mu} + \varepsilon^2\kappa^2\beta_2\mu_*/2)} \Big/ \sqrt{(\lambda + 2\mu)(1+\varepsilon^2\kappa^2\beta_2)}$, quasi-transverse wave $\zeta_2 = \sqrt{\mu} \Big/ \sqrt{(\lambda + 2\mu)}$.

For $\alpha = \pi/2$:

Quasi-longitudinal wave $\zeta_1 = 1$,

quasi-transverse wave $\zeta_2 = \sqrt{(\tilde{\mu} + \varepsilon^2\kappa^2\beta_2\mu_*)} \Big/ \sqrt{(\lambda + 2\mu)(1+\varepsilon^2\kappa^2\beta_2)}$.

For any α solution of this equation will be sought in the approximation

$$\zeta^2 = {\zeta_0}^2 + {\zeta_*}^2\varepsilon^2 + o(\varepsilon^2)$$

Zero by ε approximation for ${\zeta_0}^2$ is obtained from equation:

$$ {\zeta_0}^4 - \left(1 + \frac{\tilde{\mu}}{(\lambda+2\mu)}\right){\zeta_0}^2 + \frac{\tilde{\mu}}{(\lambda+2\mu)} + \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{(\mu-\tilde{\mu})}{(\lambda+2\mu)}\sin^2 2\alpha = 0 $$

Values ${\zeta_0}^2$, which correspond to quasi-longitudinal and quasi-transverse waves in layered medium are:

$$ {\zeta_0}^2 = 0.5\left(1 + \frac{\tilde{\mu}}{(\lambda+2\mu)} \pm \sqrt{\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2} + 2\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{(\mu-\tilde{\mu})}{(\lambda+2\mu)}\cos 4\alpha + \frac{(\mu-\tilde{\mu})^2}{(\lambda+2\mu)^2}}\right) $$

Correction coefficient ${\zeta_*}^2$ is:

$$ {\zeta_*}^2 = \beta_2\kappa^2({\zeta_0}^2 - \cos^2 2\alpha)\left(\frac{\mu_*}{(\lambda+2\mu)}\sin^2\alpha - {\zeta_0}^2\right) \Big/ \left(2{\zeta_0}^2 - \left(1 + \frac{\tilde{\mu}}{(\lambda+2\mu)}\right)\right) $$

Taking into account the expression for ${\zeta_0}^2$ get the desired values:

$$ \zeta^2 \approx {\zeta_0}^2 \pm \kappa^2\varepsilon^2\beta_2({\zeta_0}^2 - \cos^2 2\alpha)\left({\zeta_0}^2 - \frac{\mu_*}{(\lambda+2\mu)}\sin^2\alpha\right) \Big/ \sqrt{\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2} + 2\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{(\mu-\tilde{\mu})}{(\lambda+2\mu)}\cos 4\alpha + \frac{(\mu-\tilde{\mu})^2}{(\lambda+2\mu)^2}} $$

$$\zeta \approx \zeta_0 \left(1 \pm \kappa^2 \varepsilon^2 \beta_2 (\zeta_0{}^2 - \cos^2 2\alpha) \left(\zeta_0{}^2 - \frac{\mu_*}{(\lambda+2\mu)} \sin^2 \alpha \right) / 2\zeta_0{}^2 \sqrt{\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2} + 2\frac{(\lambda+\mu)}{(\lambda+2\mu)} \frac{(\mu-\tilde{\mu})}{(\lambda+2\mu)} \cos 4\alpha + \frac{(\mu-\tilde{\mu})^2}{(\lambda+2\mu)^2}} \right)$$

From these formulas easy to see that the propagation velocities of harmonic waves have small dispertion (член ~ $\kappa^2 \varepsilon^2$) and dependance on direction of wave propagation α.

Let's investigate the limiting cases of these formulas $\varepsilon \to 0$ ($\mu_\varepsilon \to \tilde{\mu}$),case of complete adhesion layers (homogeneous elastic medium) $k \to \infty$ ($\tilde{\mu} \to \mu$) and case of an ideal slip layers $k \to 0$ ($\tilde{\mu} \to 0$).

Quasi-longitudinal waves (sign + in formulas for ζ_0 and ζ).

In this case $\zeta \to \zeta_0$ for $\varepsilon \to 0$.

For $k \to \infty$ $\zeta \to 1$ ($c \to c_1$) – elastic longitudinal wave in an isotropic medium.

For $k \to 0$

$$\zeta_0{}^2 \to 0.5 \left(1 + \sqrt{\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2} + \frac{2(\lambda+\mu)\mu}{(\lambda+2\mu)^2} \cos 4\alpha + \frac{\mu^2}{(\lambda+2\mu)^2}} \right)$$

For $\alpha = 0, \pi/2$ (waves along and cross the layers)

$\zeta_0 \to 1$, $c \to c_1$

For $\alpha = \pi/4$ (waves at an angle to the direction of layers, the minimum speed)

$$\zeta_0 \to \sqrt{\frac{(\lambda+\mu)}{(\lambda+2\mu)}}, \ c \to \sqrt{\frac{(\lambda+\mu)}{(\lambda+2\mu)}} c_1$$

quasi-transverse waves (sign – in formulas for ζ_0 and ζ).

In this case $\zeta \to \zeta_0$ for $\varepsilon \to 0$.

For $k \to \infty$ $\zeta \to c_2 / c_1$ ($c \to c_2$) – elastic shear wave in an isotropic medium.

For $k \to 0$

$$\zeta_0{}^2 \to 0.5 \left(1 - \sqrt{\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2} + \frac{2(\lambda+\mu)\mu}{(\lambda+2\mu)^2} \cos 4\alpha + \frac{\mu^2}{(\lambda+2\mu)^2}} \right)$$

For $\alpha = 0, \pi/2$ (waves along and cross the layers)

$\zeta_0 \to 0$, $c \to 0$

For $\alpha = \pi / 4$ (waves at an angle to the direction of layers, the maximum speed)

$\zeta_0 \to c_2 / c_1$, $c \to c_2$

Dependence of the velocities of quasi-longitudinal waves and quasi-transverse waves from the coupling coefficients of layers k are shown in Fig. 1-5. The upper series of curves in these figures correspond to quasilongitudinal wave and the lower series correspond quasitransverse waves for different values ε / l =0.5, 0.3, 0.1. To the dimensionless elastic moduli of the layers are given values $\lambda / (\lambda + 2\mu) = \mu / (\lambda + 2\mu) = 1/3$.

Above each curve is shown the angle α =0, 30^0, 45^0, 60^0, 90^0 of wave propagation direction. For α =0, 45^0, 90^0 the curves in Fig. 1, 3, 6 are described by exact formulas, given above. For other values of α biquadratic equation for $\zeta = c / c_1$ does not have exact formula, only graphic representation (Fig. 2, 4).

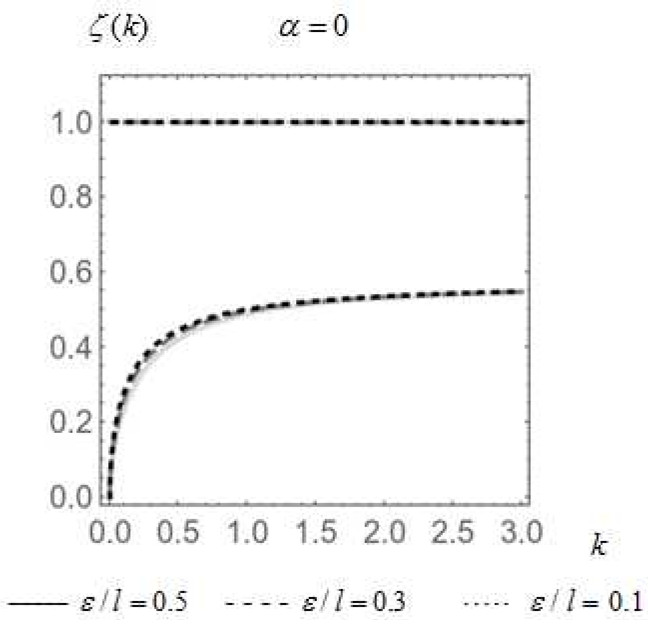
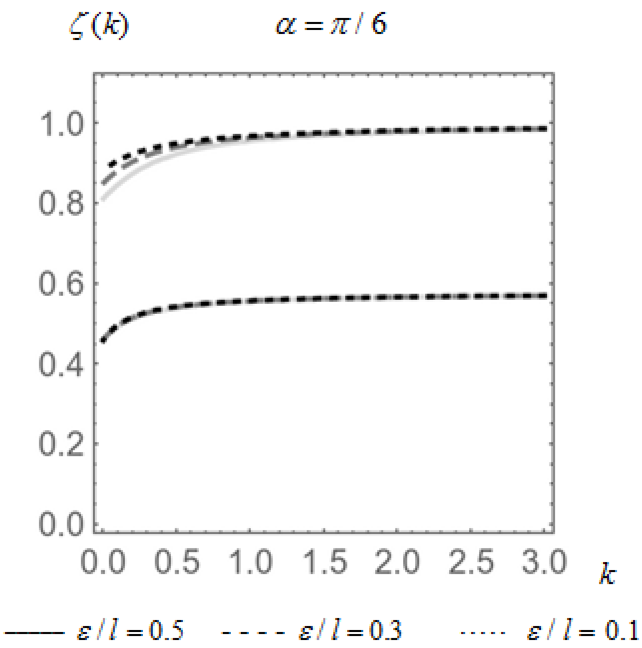
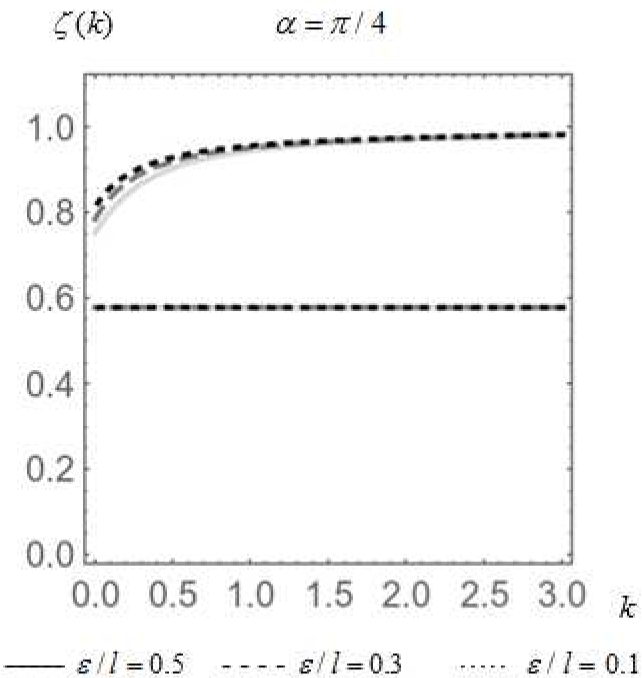


Fig.1



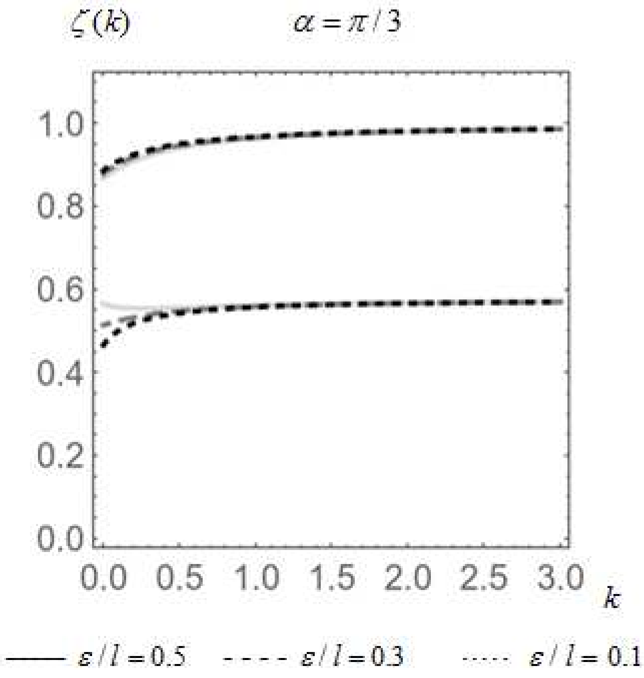
$\zeta(k)$
$\alpha = \pi / 6$
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0
k
$\varepsilon / l = 0.5$
$\varepsilon / l = 0.3$
$\varepsilon / l = 0.1$

Fig. 2



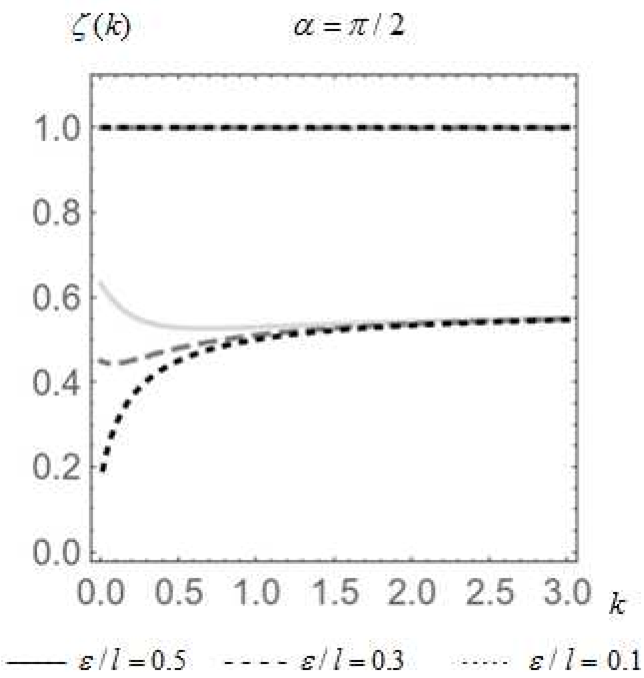
$\zeta(k)$
$\alpha = \pi / 4$
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0
k
$\varepsilon / l = 0.5$
$\varepsilon / l = 0.3$
$\varepsilon / l = 0.1$

Fig. 3



$\zeta(k)$
$\alpha = \pi/3$
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0
k
$\varepsilon/l = 0.5$
$\varepsilon/l = 0.3$
$\varepsilon/l = 0.1$

Fig. 4



$\zeta(k)$
$\alpha = \pi/2$
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0
k
$\varepsilon/l = 0.5$
$\varepsilon/l = 0.3$
$\varepsilon/l = 0.1$

Fig. 5

Из этих графиков виден уровень дисперсии плоских волн в среде (при небольших значениях коэффициентах связи слоев) для разных направлений распространения и ее зависимость от отношения толщины слоя к длине волны

From these graphs it can be seen the dispersion level of the plane waves (for small values of the layer coupling coefficients) for different directions of propagation and its dependence on the ratio of layer thickness to the wavelength ε / l. It is possible to conclude that the dispersion is observed only for dimensionless coupling coefficients $k/(\lambda+2\mu)$<0.7. It is mostly significant for directions α =90^0 (along layers) for quasi-transverse waves), see. Fig. 5, lower series of curves.

3.2. Superficial Rayleigh waves.

Consider the surface waves on the boundary of a layered half-space $-\infty < x_3 \le 0$, $-\infty < x_1 < \infty$ (flat task). The system of equations for the displacements of the layered medium with a slip at the interlayer boundaries derived earlier:

$$\left((\lambda+2\mu)w_{1,11} - \rho w_{1,tt}\right) + \lambda w_{3,13} + V_{,3} = 0$$

$$\lambda w_{1,13} + \left((\lambda+2\mu)w_{3,33} - \rho w_{3,tt}\right) + V_{,1} = 0$$

$$w_{1,3} + w_{3,1} - U = 0$$

$$\tilde{\mu} u - \varepsilon^2 \beta_2 (\mu_* u_{,11} + \rho u_{,tt}) - V = 0$$

Boundary conditions for $x_3 = 0$:

$$\sigma_{33} = (\lambda+2\mu)w_{3,3} + \lambda w_{1,1} = 0$$

$$\sigma_{13} = \mu(w_{1,3} + w_{3,1}) = 0$$

For $x_3 \to -\infty$: $w_1 \to 0$, $w_3 \to 0$.

We represent the solution of the problem in the form of a surface wave, γ>0:

$$w_1 = A e^{\gamma x_3} e^{i(\kappa_1 x_1 - \omega t)}$$

$$w_3 = B e^{\gamma x_3} e^{i(\kappa_1 x_1 - \omega t)}$$

Substituting these expressions in to the system of differential equations, we obtain a homogeneous system of algebraic equations:

$$\left(\mu_\varepsilon\gamma^2-\kappa_1^2\Delta_1\right)A+(\lambda+\mu_\varepsilon)\gamma i\kappa_1 B=0$$

$$-\kappa_1^2(\lambda+\mu_\varepsilon)\gamma A+\left((\lambda+2\mu)\gamma^2-\kappa_1^2\Delta_{2\varepsilon}\right)i\kappa_1 B=0$$

Here the following new combinations of coefficients are introduced:

$$\mu_\varepsilon=\tilde{\mu}+\varepsilon^2\beta_2\kappa_1^2\Delta_*,\quad \Delta_*=\mu_*-\rho c^2,\quad \Delta_1=\lambda+2\mu-\rho c^2$$

$$\Delta_{2\varepsilon}=\Delta_2+\varepsilon^2\beta_2\kappa_1^2\Delta_*,\quad \Delta_2=\tilde{\mu}-\rho c^2,$$

where $c=\omega/\kappa_1$ is the phase velocity of the desired surface wave..

Condition for the solvability of this system gives the biquadratic equation to determine the index γ:

$$(\lambda+2\mu)\mu_\varepsilon\gamma^4-\kappa_1^2\gamma^2\left(\mu_\varepsilon\Delta_{2\varepsilon}+(\lambda+2\mu)\Delta_1-(\lambda+\mu_\varepsilon)^2\right)+\kappa_1^4\Delta_1\Delta_{2\varepsilon}=0$$

From this equation, we find two positive solutions $\gamma_{1,2}>0$:

$$\gamma_{1,2}{}^2=\frac{\kappa_1^2\left\{\left(\mu_\varepsilon\Delta_{2\varepsilon}+(\lambda+2\mu)\Delta_1-(\lambda+\mu_\varepsilon)^2\right)\pm\sqrt{\left(\mu_\varepsilon\Delta_{2\varepsilon}+(\lambda+2\mu)\Delta_1-(\lambda+\mu_\varepsilon)^2\right)^2-4(\lambda+2\mu)\mu_\varepsilon\Delta_1\Delta_{2\varepsilon}}\right\}}{2(\lambda+2\mu)\mu_\varepsilon}$$

Thus, taking into account this fact, the solution of this problem takes view:

$$w_1=A_1e^{\gamma_1x_3}e^{i(\kappa_1x_1-\omega t)}+A_2e^{\gamma_2x_3}e^{i(\kappa_1x_1-\omega t)}$$

$$w_3=B_1e^{\gamma_1x_3}e^{i(\kappa_1x_1-\omega t)}+B_2e^{\gamma_2x_3}e^{i(\kappa_1x_1-\omega t)}$$

$$i\kappa_1B_{1,2}=\kappa_1^2\frac{(\lambda+\mu_\varepsilon)\gamma_{1,2}A_{1,2}}{\left((\lambda+2\mu)\gamma_{1,2}{}^2-\kappa_1^2\Delta_{2\varepsilon}\right)}$$

Substituting these solutions into the boundary conditions at $x_3=0$ and get the system of equations:

$$\gamma_1A_1+\gamma_2A_2+i\kappa_1B_1+i\kappa_1B_2=0$$

$$-\lambda\kappa_1{}^2A_1-\lambda\kappa_1{}^2A_2+(\lambda+2\mu)\gamma_1i\kappa_1B_1+(\lambda+2\mu)\gamma_2i\kappa_1B_2=0$$

The amplitudes B_1 and B_2 can be eliminated, then we have two homogeneous algebraic equations for amplitudes A_1 and A_2. For farther simplification instead of $\gamma_{1,2} > 0$ introduce values $\eta_{1,2}$ from relation $\eta_{1,2} = \gamma_{1,2} / \kappa_1$. These values are defined by formulas:

$$\eta_{1,2}{}^2 = \frac{\mu_\varepsilon \Delta_{2\varepsilon} + (\lambda + 2\mu)\Delta_1 - (\lambda + \mu_\varepsilon)^2 \pm \sqrt{\left(\mu_\varepsilon \Delta_{2\varepsilon} + (\lambda + 2\mu)\Delta_1 - (\lambda + \mu_\varepsilon)^2\right)^2 - 4(\lambda + 2\mu)\mu_\varepsilon \Delta_1 \Delta_{2\varepsilon}}}{2(\lambda + 2\mu)\mu_\varepsilon}$$

Homogeneous system of equations for amplitudes A_1 and A_2 is:

$$\eta_1 \left(1 + \frac{(\lambda + \mu_\varepsilon)}{\left((\lambda + 2\mu)\eta_1{}^2 - \Delta_{2\varepsilon}\right)} \right) A_1 + \eta_2 \left(1 + \frac{(\lambda + \mu_\varepsilon)}{\left((\lambda + 2\mu)\eta_2{}^2 - \Delta_{2\varepsilon}\right)} \right) A_2 = 0$$

$$\left(\frac{(\lambda + 2\mu)(\lambda + \mu_\varepsilon)\eta_1{}^2}{\left((\lambda + 2\mu)\eta_1{}^2 - \Delta_{2\varepsilon}\right)} - \lambda \right) A_1 + \left(\frac{(\lambda + 2\mu)(\lambda + \mu_\varepsilon)\eta_2{}^2}{\left((\lambda + 2\mu)\eta_2{}^2 - \Delta_{2\varepsilon}\right)} - \lambda \right) A_2 = 0$$

For solvability the determinant of this system matrix should be equal to zero. This gives the equation for phase velocity of surface wave $c = \omega / \kappa_1$:

$$4(\lambda + \mu)\eta_1\eta_2{}^2 - \eta_2(1 + \eta_2{}^2)\left((\lambda + 2\mu)\eta_1{}^2 + \lambda\eta_2{}^2\right) -$$

$$- \frac{\Delta\mu_\varepsilon}{\mu}\left\{ \eta_1\left((\lambda + 2\mu)\eta_2{}^2 + \lambda\right) + \eta_2(1 + \eta_2{}^2)\left((\lambda + 2\mu)\eta_1{}^2 + \lambda\right) \right\} = 0$$

Here $\Delta\mu_\varepsilon = \mu - \mu_\varepsilon$.

Again investigate limiting cases of these formulas for $\varepsilon \to 0$ ($\mu_\varepsilon \to \tilde{\mu}$).

In such cases

$$\eta_{1,2}{}^2 = \frac{\tilde{\mu}\Delta_2 + (\lambda + 2\mu)\Delta_1 - (\lambda + \tilde{\mu})^2 \pm \sqrt{\left(\tilde{\mu}\Delta_2 + (\lambda + 2\mu)\Delta_1 - (\lambda + \tilde{\mu})^2\right)^2 - 4(\lambda + 2\mu)\tilde{\mu}\Delta_1\Delta_2}}{2(\lambda + 2\mu)\tilde{\mu}}$$

The equation for the velocity of surface wave will be

$$4(\lambda+\mu)\eta_1\eta_2{}^2-\eta_2(1+\eta_2{}^2)\left((\lambda+2\mu)\eta_1{}^2+\lambda\eta_2{}^2\right)-$$

$$-\frac{\mu}{(k+\mu)}\left\{\eta_1\left((\lambda+2\mu)\eta_2{}^2+\lambda\right)+\eta_2(1+\eta_2{}^2)\left((\lambda+2\mu)\eta_1{}^2+\lambda\right)\right\}=0$$

Case of complete adhesion layers (homogeneous elastic medium) $k\to\infty$ *(* $\tilde{\mu}\to\mu$ *)*

In this case:

$$\eta_1{}^2=1-c^2/c_1^2,\quad \eta_2{}^2=1-c^2/c_2^2$$

$$4(\lambda+\mu)\eta_1\eta_2-(1+\eta_2{}^2)\left((\lambda+2\mu)\eta_1{}^2+\lambda\eta_2{}^2\right)=0$$

From this we come to classical Reyleigh wave:

$$4\sqrt{1-c^2/c_1^2}\sqrt{1-c^2/c_2^2}-(2-c^2/c_2^2)^2=0$$

Case of ideal slip of layers $k\to 0$ *(* $\tilde{\mu}\to 0$ *)*

In this case, supposing that μ_ε is a small parameter, get:

$$\eta_1^2\sim\frac{4\mu(\lambda+\mu)-(\lambda+2\mu)\rho c^2}{(\lambda+2\mu)\mu_\varepsilon}$$

$$\eta_2^2\sim\frac{(\lambda+2\mu-\rho c^2)(\mu_\varepsilon-\rho c^2)}{4\mu(\lambda+\mu)-(\lambda+2\mu)\rho c^2}$$

$$(3\lambda+2\mu)\eta_1\eta_2{}^2-2(\lambda+2\mu)\eta_1{}^2\eta_2(1+\eta_2{}^2)-\lambda\eta_2(1+\eta_2{}^2)^2-\lambda\eta_1=0$$

The graphs of the dependence of dimensionless velocity of surface wave c/c_2 on the layer coupling coefficients k are shown in Fig. 6 for various values of ratio ε/l =0.5, 0.3, 0.1. As in the previous case the wave number $\kappa_1=2\pi/l$, where l is a length of harmonic surface wave. Also here $\varepsilon\kappa_1=2\pi(\varepsilon/l)$, $\varepsilon^2\kappa_1{}^2=4\pi^2(\varepsilon/l)^2$. This solution becomes close to classical Reyleigh root for the range of coupling coefficient values $k/(\lambda+2\mu)>1.5\div 2$.

Behavior of these curves is very similar to the behavior of lower series of curves in Fig. 5 (*quasi-transverse waves*) for waves propagating along layers (α =90^0) and in close directions. For classical Reyleigh waves, as known, $c_R/c_2\approx 0.9$, the same relation takes place and in case under consideration for ratio of surface wave velocity to the velocity of quasi-transverse waves.

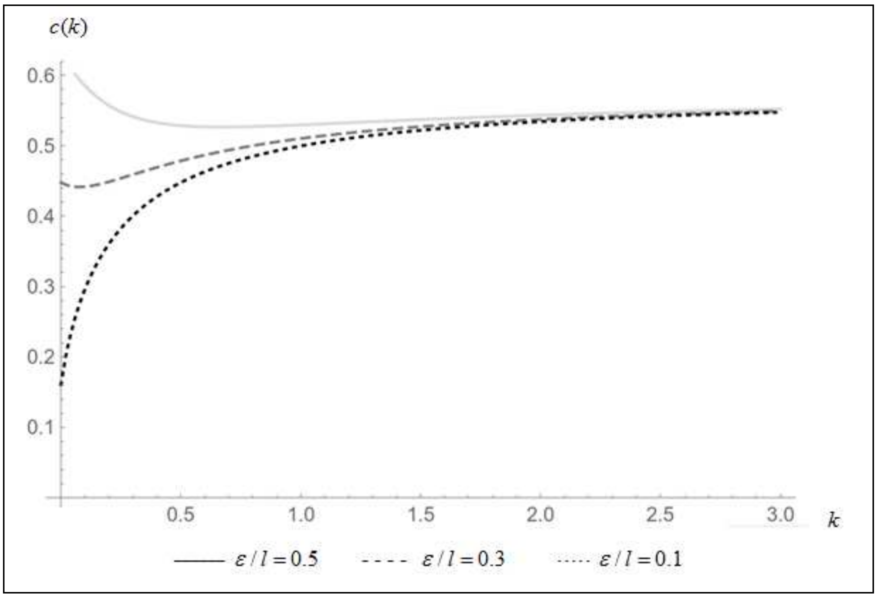


Fig. 6

In conclusion note, that the limits of applicability of the obtained asymptotic theory are not precisely defined. Rather arbitrary in the calculation was adopted the upper limit of the small parameter ε / l =0.5. Nonetheless, for the layer coupling coefficients starting with values $k / (\lambda + 2\mu) > 0.7$, the calculations give close results for propagation velocities of quasi-longitudinal, quasi-transverse and surface waves for the whole range of wave length ε / l <0.5.

Suppose, that derived refined theory is possible to use for investigation of seismic waves transformation during their exit to terrestrial surface of layered rock massifs, accounting slip shifts on contact interlayer boundaries. Also this theory may be useful for description of deformation in composite layered materials with intermediate soft sublayers (rubber sublayers betweb metallic layers).

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Conclusions

На основе асимптотического метода осреднения получены континуальные уравнения слоистой среды с проскальзыванием с учетом членов второго порядка по малому параметру толщины слоя. Использовано линейное условие проскальзывания, связывающее скачки касательных смещений на контактных границах и касательные напряжения. Исследованы волновые свойства полученной системы уравнений, получены дисперсионные соотношения для гармонических волн. Построено решение задачи о поверхностной волне типа Рэлея на границе упругого слоистого полупространства.

On the basis of the asymptotic method of averaging obtained the system of refined equations for layered continuum medium with interlayer slipping. The theory takes into account terms of the second order regarding the layer thickness as small parameter. Using linear slip condition between the tangential jumps of displacements at the contact interlayer boundaries and shear stresses. The wave properties of the resulting system of equations, including dispersion relations for harmonic waves are investigated. The solution of the problem of Rayleigh surface waves on the boundary of an elastic layered half-space is found.

The created refined theory of layered media may be useful in research of seismic waves propagation in layered rock massifs and in study of deformation properties of composite layered materials.

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