A Refined Theory of the Layered Medium with the Slip at the Interface

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Abstract The equations for layered medium with slippage are obtained using the asymptotic method of homogenisation. The terms of second order respectively the small parameter of layer thickness are taken into account. The linear slip condition defines the dependence between the tangential jumps of displacements at the contact boundary and the shear stresses. The derived equations introduce asymptotically complete generalization of some models of layered media, which are based on the engineering approach or approximate hypotheses about the nature of the inter-layer deformation. Such generalized models are needed in the study of static and dynamic deformations of layered rock media. The wave properties of the resulting system of equations and dispersion relations for harmonic waves are described. The propagation of Rayleigh surface waves along the elastic layered half-plane boundary is investigated.

1 Introduction

The interest to the problem of propagation and transformation of waves in layered media is initiated by the seismology and engineering geophysics. As a rule the seismicity is observed in rock regions. Often these rocks contain regular grid of cracks which can be considered as layered structures. Classical studies of wave fields in such media usually are based on assumption of continuity of displacement fields. But for rather strong seismic actions the possibility of tangential displacement jumps at the inter-layer boundaries should be taken in to account. For long time actions it needs to

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use the "averaged" models of structured continuum media because of impossibility to trace deformations of each structural layer.

In our study by using asymptotic method [1, 6] the refined equations of layered medium with slippage are derived. The second order terms relatively small parameter of layer thickness are taken in to account. The linear slip relation between tangential displacement jumps at inter-layer boundaries and shear stresses is used. The zero order approximate equations for such media has been derived earlier in [3, 4]. The proposed here new equations represent complete generalization of layered media models [5, 7], which are based on engineering approaches or on approximate hypothesizes about layer deformations. Such generalized models are required for static and dynamic problems of rock media deformations and for dynamic wave propagation problems in geophysics. It should be noted also that the theory of layered media is suitable for description of composite materials with soft (rubber) sublayers between major more rigid (metallic) layers.

The properties of proposed refined system of equations are studied. The propagation of longitudinal, transversal and surface Rayleigh waves in layered media is investigated in refined settings.

2 Refined Equations

Consider infinite layered medium using Cartesian rectangular coordinate system (x_1, x_2, x_3). The axis x_3 is perpendicular to the planes of parallel flat boundaries between layers. Let the inter-layer boundaries have coordinates $x_3 = x^{(s)} = s\varepsilon$ $(s = 0, \pm 1, \pm 2, ...)$, where constant layer thickness $\varepsilon \ll 1$ is a small parameter. To say more exactly, the relation $\varepsilon / l \ll 1$ should be valid, here l is the size of distributed load application range, for instance, wave length in the processes under consideration. In such case all spatial values should be made dimensionless using this value l.

Assume that layer boundaries are always compressed and the following conditions are valid

$$\sigma_{33} < 0,\ [u_3] = [\sigma_{\gamma 3}] = [\sigma_{33}] = 0$$

Here $\sigma_{\gamma 3} = k_*[u_\gamma]$ is linear slippage of Winkler type, $k_*\varepsilon = k = O(1)$. Square brackets $[f] = f|_{x^{(s)}+0} - f|_{x^{(s)}-0}$ designate the jump of a value f at inter-layer boundary. Such conditions are valid approximately if the soft sublayers of thickness δ $(\delta/\varepsilon \ll 1)$ with small shear modulus μ_δ are present between layers. In this case we have

$$\sigma_{\gamma 3} = k[u_\gamma]/\varepsilon = \frac{k\delta}{\varepsilon}\frac{[u_\gamma]}{\delta} = \mu_\delta \frac{[u_\gamma]}{\delta}$$

Here $[u_\gamma]/\delta$ is shear deformation of soft sublayer. In this case $\mu_\delta = k\delta/\varepsilon$ or vise versa $k = \mu_\delta \varepsilon/\delta$. It is possible to say that is inter-layer shear connection coefficient. The layers themselves are elastic isotropic and subjected to Hooke's law

$$x_3 \neq x^{(s)} : \ \sigma_{ij,j} - \rho u_{i,tt} = 0, \ \sigma_{ij} = C_{ijkl} u_{k,l}$$

Here the elastic moduli tensor is

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

According to the method of asymptotic homogenisation [1] let's introduce fast variable $\xi = x_3/\varepsilon$. According to [1] assume that $u_k = u_k(x_l, \xi, t)$ is a function, which is smooth regarding slow variables x_l and continuous regarding fast variable ξ, excluding points $\xi^{(s)} = x^{(s)}/\varepsilon$, where it may have jumps of first kind. Besides, along ξ the displacement is 1-periodic $[[u_i]] = u_i|_{\xi^{(s)}+1/2} - u_i|_{\xi^{(s)}-1/2} = 0$. Accounting such choice of variables and the differentiation rule for complex functions, the system of equations for cell of periodicity $x^{(s)} - 1/2 \leq x_3 \leq x^{(s)} + 1/2$, $-1/2 \leq \xi \leq 1/2$ may be rewritten as

$$\varepsilon^{-2} C_{i3k3} u_{k,\xi\xi} + \varepsilon^{-1}(C_{ijk3} u_{k,j\xi} + C_{i3kl} u_{k,l\xi}) + C_{ijkl} u_{k,lj} - \rho u_{i,tt} = 0$$

where $x_3 \neq x^{(s)}, \ \xi \neq 0$. At $x_3 = x^{(s)}, \ \xi = 0$ we use the contact conditions

$$\varepsilon^{-1} C_{33k3} u_{k,\xi} + C_{33kl} u_{k,l} < 0$$

$$[u_3] = 0, [\varepsilon^{-1} C_{i3k3} u_{k,\xi} + C_{i3kl} u_{k,l}] = 0, \ \ \varepsilon^{-1} C_{\gamma 3k3} u_{k,\xi} + C_{\gamma 3kl} u_{k,l} = k_* [u_\gamma]$$

The conditions of 1-periodicity are

$$[[u_i]] = u_i|_{\xi+1/2} - u_i|_{\xi-1/2} = 0$$

Here and farther greek indices (β,γ) take values 1 and 2, latine indices take values 1, 2, 3. The displacements are represented as asymptotic series regarding small parameter ε:

$$u_i = u_i^{(0)}(x_k, \xi, t) + \varepsilon u_i^{(1)}(x_k, \xi, t) + \varepsilon^2 u_i^{(2)}(x_k, \xi, t) + \varepsilon^3 u_i^{(3)}(x_k, \xi, t) + \cdots$$

Introduce the operation of averaging $\langle f \rangle$ for the function of fast variable ξ, which will be often used farther: $\langle f \rangle = \int_{-1/2}^{1/2} f d\xi$. Displacement approximations should satisfy the additional condition $\left\langle u_k^{(n)} \right\rangle = 0$ [1].

Substitute this representation into the theory of elasticity equations. Equating to zero the term with negative power ε^{-2} we get that zero approximation $u_i^{(0)}$ is independent on the fast variable ξ and $u_i^{(0)} = w_i(x_k, t)$. Equating to zero the term with negative power ε^{-1} we get that first approximation $u_i^{(1)}$ satisfies the equation $C_{i3k3}u_{k,\xi\xi}^{(1)} = 0$. The resulting system of differential equations is:

$$
\begin{aligned}
&C_{ijkl}w_{k,jl} + C_{ijk3}u_{k,j\xi}^{(1)} + (C_{i3kl}u_{k,l}^{(1)} + C_{i3k3}u_{k,\xi}^{(2)})_{,\xi} \\
&\quad + \varepsilon\left[C_{ijkl}u_{k,jl}^{(1)} + C_{ijk3}u_{k,j\xi}^{(2)} + (C_{i3kl}u_{k,l}^{(2)} + C_{i3k3}u_{k,\xi}^{(3)})_{,\xi}\right] \\
&\quad + \varepsilon^2\left[C_{ijkl}u_{k,jl}^{(2)} + C_{ijk3}u_{k,j\xi}^{(3)} + (C_{i3kl}u_{k,l}^{(3)} + C_{i3k3}u_{k,\xi}^{(4)})_{,\xi}\right] + \cdots \\
&= \rho w_{i,tt} + \varepsilon\rho u_{i,tt}^{(1)} + \varepsilon^2\rho u_{i,tt}^{(2)} + \cdots
\end{aligned}
$$

A similar representation for stress tensor components is:

$$
\sigma_{ij} = \sigma_{ij}^{(0)} + \varepsilon\sigma_{ij}^{(1)} + \varepsilon^2\sigma_{ij}^{(2)} + \cdots
$$

where $\sigma_{ij}^{(n)} = C_{ijkl}u_{k,l}^{(n)} + C_{ijk3}u_{k,\xi}^{(n+1)}$.

All approximations for stresses are 1-periodic functions of ξ. In particular, the relation $\sigma_{i3}^{(n)} = C_{i3kl}u_{k,l}^{(n)} + C_{i3k3}u_{k,\xi}^{(n+1)}$ and conditions $[\sigma_{i3}^{(n)}] = 0$, $[[\sigma_{i3}^{(n)}]] = 0$ are valid. It is easy to see that $\left\langle \sigma_{i3}^{(n),\xi} \right\rangle = 0$.

Accounting the terms of definite order of ε, applying the averaging operation $\langle f \rangle$ and excluding the dependence on fast variable ξ, we get the model of a homogenised layered medium with slippage of Winkler type.

Let's derive the refined theory of second order. For this in the system of equations we keep the terms of order ε^2. Applying averaging operation $\langle\rangle$ for periodicity cell to the system of equations we get the following:

$$
C_{ijkl}w_{k,jl} + C_{ijk3}\left\langle u_{k,\xi}^{(1)} \right\rangle_{,j} + \varepsilon C_{ijk3}\left\langle u_{k,\xi}^{(2)} \right\rangle_{,j} + \varepsilon^2 C_{ijk3}\left\langle u_{k,\xi}^{(3)} \right\rangle_{,j} = \rho w_{i,tt}
$$

It is the final refined system of equations for layered medium with slippage. For complete formulations it needs to find the functions $\left\langle u_{k,\xi}^{(n)} \right\rangle$ (n = *1, 2, 3*), which participate in the system. Every function $u_i^{(n)}(x_k, \xi, t)$ (n = *1, 2, 3*) is found from the appropriate task in periodicity cell $(-1/2 \le \xi \le 1/2)$ [1], which is formulated by equating to zero the sum of terms of definite order ε^{n-1} in asymptotic system of equations. Additional conditions for these functions can be received by reformulating the contact inter-layer conditions for each function: conditions of 1-periodicity $[[u_i^{(n)}]] = 0$ and conditions $\left\langle u_i^{(n)} \right\rangle = 0$. Let's formulate these three tasks for the cell $(-1/2 \le \xi \le 1/2)$.

2.1 Task in Cell for n = 1

At $|\xi| < 1/2$: $C_{i3k3}u^{(1)}_{k,\xi\xi} = 0$.

At $\xi = 0$: $[C_{i3k3}u^{(1)}_{k,\xi}] = 0$, $[u^{(1)}_3] = 0$, $k[u^{(1)}_\gamma] = C_{\gamma 3kl}w_{k,l} + C_{\gamma 3k3}u^{(1)}_{k,\xi}$.

Additional conditions are: $[[u^{(1)}_i]] = 0$, $\left\langle u^{(1)}_i \right\rangle = 0$.

Dropping details, published in [2], write the solution of task 1 on the periodicity cell:

$u^{(1)}_\gamma = \phi_\gamma(\xi - sign\xi/2), u^{(1)}_3 = 0$, where $\phi_\gamma = -\tau_\gamma/(k+\mu)$, $\tau_\gamma = \mu(w_{\gamma,3}+w_{3,\gamma})$.

The derivatives needed for averaging are:

$$u^{(1)}_{3,\xi} = 0, u^{(1)}_{\gamma,\xi} = \phi_\gamma,$$
$$\left\langle u^{(1)}_{3,\xi} \right\rangle = 0, \left\langle u^{(1)}_{\gamma,\xi} \right\rangle = \phi_\gamma$$

2.2 Task in Cell for n = 2

At $|\xi| < 1/2$ have

$$C_{ijkl}w_{k,jl} + C_{ijk3}u^{(1)}_{k,j\xi} + (C_{i3kl}u^{(1)}_{k,l} + C_{i3k3}u^{(2)}_{k,\xi})_{,\xi} = \rho w_{i,tt}$$

Averaging this differential equation and accounting that

$$\left\langle (C_{i3kl}u^{(1)}_{k,l} + C_{i3k3}u^{(2)}_{k,\xi})_{,\xi} \right\rangle = 0$$

and that the rest terms of this equation do not depend on ξ, we get its simple consequence:

$$C_{i3k3}u^{(2)}_{k,\xi\xi} = -C_{i3kl}u^{(1)}_{k,\xi l}$$

At $\xi = 0$ have

$$[C_{i3k3}u^{(2)}_{k,\xi}] = -[C_{i3kl}u^{(1)}_{k,l}],\ [u^{(2)}_3] = 0,\ k[u^{(2)}_\gamma] = C_{\gamma 3kl}u^{(1)}_{k,l} + C_{\gamma 3k3}u^{(2)}_{k,\xi}$$

Additional conditions are $[[u^{(2)}_i]] = 0$, $\left\langle u^{(2)}_i \right\rangle = 0$.

Dropping details (see in [2]), write the solution of task 2 on periodicity cell

$$u^{(2)}_\gamma = -\psi_\gamma(\xi^2 - \xi sign\xi + 1/6)/2, u^{(2)}_3 = -\psi_3(\xi^2 - \xi sign\xi + 1/6)/2$$

where $\psi_\gamma = \phi_{\gamma,3}$, $\psi_3 = \lambda\phi_{\beta,\beta}/(\lambda + 2\mu)$

Derivatives needed for averaging are

$$u^{(2)}_{\gamma,\xi} = -\psi_\gamma(\xi - sign\xi/2), u^{(2)}_{3,\xi} = -\psi_3(\xi - sign\xi/2), \left\langle u^{(2)}_{3,\xi}\right\rangle = 0, \left\langle u^{(2)}_{\gamma,\xi}\right\rangle = 0$$

Hence second approximations for displacements are absent in refined system of equations.

2.3 Task in Cell for n = 3

At $|\xi| < 1/2$ have

$$C_{i3k3}u^{(3)}_{k,\xi\xi} = -C_{ijkl}u^{(1)}_{k,jl} - C_{i3kl}u^{(2)}_{k,\xi l} - C_{ijk3}u^{(2)}_{k,\xi j} + \rho u^{(1)}_{i,tt}$$

At $\xi = 0$ have

$$[C_{i3k3}u^{(3)}_{k,\xi}] = -[C_{i3kl}u^{(2)}_{k,l}],\ [u^{(3)}_3] = 0,\ k[u^{(3)}_\gamma] = C_{\gamma 3kl}u^{(2)}_{k,l} + C_{\gamma 3k3}u^{(3)}_{k,\xi}.$$

Additional conditions are $[[u^{(3)}_i]] = 0, \left\langle u^{(3)}_i\right\rangle = 0$

Consider solution for cases $i = \gamma$. The elasticity moduli tensor is

$$C_{ijkl}u^{(1)}_{k,jl} = C_{\gamma j\beta l}u^{(1)}_{\beta,jl}$$

$$= (\lambda\delta_{\gamma j}\delta_{\beta l} + \mu\delta_{\gamma\beta}\delta_{jl} + \mu\delta_{\gamma l}\delta_{j\beta})u^{(1)}_{\beta,jl} = (\lambda+\mu)u^{(1)}_{\beta,\beta\gamma} + \mu u^{(1)}_{\gamma,ll}$$

$$(C_{\gamma 3kl} + C_{\gamma lk3})u^{(2)}_{k,\xi l} = \left((\lambda+\mu)\delta_{\gamma l}\delta_{3k} + 2\mu\delta_{\gamma k}\delta_{3l}\right)u^{(2)}_{k,\xi l} = (\lambda+\mu)u^{(2)}_{3,\xi\gamma} + 2\mu u^{(2)}_{\gamma,\xi 3}$$

Task equation for $|\xi| < 1/2$ is

$$u^{(3)}_{\gamma,\xi\xi} = u^{(1)}_{\gamma,ll} - (\lambda+\mu)u^{(1)}_{\beta,\beta\gamma}/\mu - 2u^{(2)}_{\gamma,\xi 3} - (\lambda+\mu)u^{(2)}_{3,\xi\gamma}/\mu + \rho u^{(1)}_{i,tt}/\mu$$

At $\xi = 0$ have following conditions

$$[u^{(3)}_{\gamma,\xi}] = -[u^{(2)}_{\gamma,3} + u^{(2)}_{3,\gamma}] = 0$$
$$k[u^{(3)}_\gamma] = \mu(u^{(2)}_{\gamma,3} + u^{(2)}_{3,\gamma} + u^{(3)}_{\gamma,\xi})$$
$$[[u^{(3)}_\gamma]] = 0, \left\langle u^{(3)}_\gamma\right\rangle = 0$$

The equation may be rewritten as

$$u^{(3)}_{\gamma,\xi\xi} = \chi_\gamma\ (\xi - sign\xi/2)$$

where $\chi_\gamma = -\phi_{\gamma,ll} - (\lambda+\mu)\phi_{\beta,\beta\gamma}/\mu + 2\psi_{\gamma,3} + (\lambda+\mu)\psi_{3,\gamma}/\mu + \rho\phi_{\gamma,tt}/\mu$.

Integrating and accounting conditions for $\xi = 0$, we get [2]

$$u^{(3)}_{\gamma,\xi} = \chi_\gamma \left(\xi^2 - \xi sign\xi\right)/2 + \left(k\chi_\gamma + \mu\psi_{\gamma,3} + \mu\psi_{3,\gamma}\right)/(k+\mu)/12$$

Finally the expression for refined derivative is

$$\left\langle u^{(3)}_{\gamma,\xi}\right\rangle = \mu\left(\phi_{\gamma,\beta\beta} + (3\lambda+2\mu)\phi_{\beta,\beta\gamma}/(\lambda+2\mu) - \rho\phi_{\gamma,tt}/\mu\right)/(k+\mu)/12$$

Now consider solution for case $i = 3$. The elasticity moduli tensor is

$$C_{3jkl}u^{(1)}_{k,jl} = C_{3j\beta l}u^{(1)}_{\beta,jl} = (\lambda\delta_{3j}\delta_{\beta l} + \mu\delta_{3\beta}\delta_{jl} + \mu\delta_{3l}\delta_{j\beta})u^{(1)}_{\beta,jl} = (\lambda+\mu)u^{(1)}_{\beta,\beta 3}$$

$$(C_{33kl} + C_{3lk3})u^{(2)}_{k,\xi l} = ((\lambda+3\mu)\delta_{3l}\delta_{3k} + (\lambda+\mu)\delta_{kl})\, u^{(2)}_{k,\xi l}$$

$$= 2(\lambda+2\mu)u^{(2)}_{3,\xi 3} + (\lambda+\mu)u^{(2)}_{\beta,\xi\beta}$$

Task equation for $|\xi| < 1/2$ is

$$u^{(3)}_{3,\xi\xi} = -(\lambda+\mu)u^{(1)}_{\beta,\beta 3}/(\lambda+2\mu) - 2u^{(2)}_{3,\xi 3} - (\lambda+\mu)u^{(2)}_{\beta,\xi\beta}/(\lambda+2\mu)$$

$\xi = 0$: $[u^{(3)}_{3,\xi}] = -[u^{(2)}_{3,3}] - \lambda[u^{(2)}_{\beta,\beta}]/(\lambda+2\mu) = 0$
$u^{(2)}_3 = 0$, $[[u^{(3)}_3]] = 0$, $\left\langle u^{(3)}_3\right\rangle = 0$.
The equation may be rewritten as:

$$u^{(3)}_{3,\xi\xi} = \chi_3\,(\xi - sign\xi/2)$$

Here $\chi_3 = (\lambda+\mu)\psi_{\beta,\beta}/(\lambda+2\mu) + 2\psi_{3,3} - (\lambda+\mu)\phi_{\beta,\beta 3}/(\lambda+2\mu)$.
Integrating and accounting conditions for $\xi = 0$ we get [2]

$$u^{(3)}_{3,\xi} = \chi_3(\xi^2 - \xi sign\xi + 1/6)/2, \left\langle u^{(3)}_{3,\xi}\right\rangle = 0.$$

Finally the expressions for refined derivatives are

$$\left\langle u^{(3)}_{\gamma,\xi}\right\rangle = \frac{1}{12}\frac{\mu}{(k+\mu)}\left(\phi_{\gamma,\beta\beta} + \frac{3\lambda+2\mu}{\lambda+2\mu}\phi_{\beta,\beta\gamma} - \frac{\rho}{\mu}\phi_{\gamma,tt}\right), \left\langle u^{(3)}_{3,\xi}\right\rangle = 0.$$

3 Variants of Averaged System of Equations

Now we can formulate the refined system of equations for layered medium with slippage (latine indices *i, j, k, l* = 1, 2, 3; greek indices $\beta,\ \gamma$ = 1, 2):

$$C_{\gamma jkl}w_{k,jl} + C_{\gamma jk3}\left\langle u^{(1)}_{k,\xi}\right\rangle_{,j} + \varepsilon^2 C_{\gamma jk3}\left\langle u^{(3)}_{k,\xi}\right\rangle_{,j} = \rho w_{\gamma,tt}$$
$$C_{3jkl}w_{k,jl} + C_{3jk3}\left\langle u^{(1)}_{k,\xi}\right\rangle_{,j} + \varepsilon^2 C_{3jk3}\left\langle u^{(3)}_{k,\xi}\right\rangle_{,j} = \rho w_{3,tt}$$

Accounting the elastic moduli tensor the terms of this system of equations are written as

$$C_{\gamma jkl}w_{k,jl} = (\lambda+\mu)w_{k,k\gamma} + \mu w_{\gamma,kk}, \quad C_{3jkl}w_{k,jl} = (\lambda+\mu)w_{k,k3} + \mu w_{3,kk}$$

$$C_{\gamma jk3}\left\langle u^{(1)}_{k,\xi}\right\rangle_{,j} = C_{\gamma j\beta 3}\left\langle u^{(1)}_{\beta,\xi}\right\rangle_{,j} = \mu\phi_{\gamma,3}$$

$$C_{3jk3}\left\langle u^{(1)}_{k,\xi}\right\rangle_{,j} = C_{3j\beta 3}\left\langle u^{(1)}_{\beta,\xi}\right\rangle_{,j} = \mu\phi_{\beta,\beta}$$

$$C_{\gamma jk3}\left\langle u^{(3)}_{k,\xi}\right\rangle_{,j} = \mu\left\langle u^{(3)}_{\gamma,\xi}\right\rangle_{,3}$$

$$= \mu^2\left(\phi_{\gamma,\beta\beta 3} + (3\lambda+2\mu)\phi_{\beta,\beta\gamma 3}/(\lambda+2\mu) - \rho\phi_{\gamma,tt3}/\mu\right)/(k+\mu)/12$$

$$C_{3jk3}\left\langle u^{(3)}_{k,\xi}\right\rangle_{,j} = \left\langle u^{(3)}_{\beta,\xi}\right\rangle_{,\beta}$$

$$= \mu^2\left(4(\lambda+\mu)\phi_{\beta,\beta\alpha\alpha}/(\lambda+2\mu) - \rho\phi_{\beta,\beta tt}/\mu\right)/(k+\mu)/12$$

Finally refined system of equations is

$$\begin{aligned}
&(\lambda+\mu)w_{k,k\gamma} + \mu w_{\gamma,kk} + \mu\phi_{\gamma,3}\\
&\quad + \varepsilon^2\mu^2\left(\phi_{\gamma,\beta\beta 3} + (3\lambda+2\mu)\phi_{\beta,\beta\gamma 3}/(\lambda+2\mu) - \rho\phi_{\gamma,tt3}/\mu\right)/(k+\mu)/12 = \rho w_{\gamma,tt}\\
&(\lambda+\mu)w_{k,k3} + \mu w_{3,kk} + \mu\phi_{\beta,\beta}\\
&\quad + \varepsilon^2\mu^2\left(4(\lambda+\mu)\phi_{\beta,\beta\alpha\alpha}/(\lambda+2\mu) - \rho\phi_{\beta,\beta tt}/\mu\right)/(k+\mu)/12 = \rho w_{3,tt}
\end{aligned}$$

Remind that $\phi_\gamma = -\mu(w_{\gamma,3}+w_{3,\gamma})/(k+\mu)$. In general equations the expressions for ϕ_γ are not substituted to avoid the unnecessary complexity of formulas. It is seen that regarding spatial variables this is the system of forth order for the displacements w_k and it contains mixed time derivatives.

The system of equations is simplified for the case of ideal slipping contact between layers $k = 0$.

$$\begin{aligned}
&(\lambda+\mu)w_{k,k\gamma} + \mu w_{\gamma,kk} + \mu\phi_{\gamma,3}\\
&\quad + \varepsilon^2\mu\left(\phi_{\gamma,\beta\beta 3} + (3\lambda+2\mu)\phi_{\beta,\beta\gamma 3}/(\lambda+2\mu) - \rho\phi_{\gamma,tt3}/\mu\right)/12 = \rho w_{\gamma,tt}\\
&(\lambda+\mu)w_{k,k3} + \mu w_{3,kk} + \mu\phi_{\beta,\beta}\\
&\quad + \varepsilon^2\mu\left(4(\lambda+\mu)\phi_{\beta,\beta\alpha\alpha}/(\lambda+2\mu) - \rho\phi_{\beta,\beta tt}/\mu\right)/12 = \rho w_{3,tt}\\
&\phi_\gamma = -(w_{\gamma,3}+w_{3,\gamma})
\end{aligned}$$

Separately we formulate plane (2D) dynamic system of equations

$$
\begin{aligned}
&(\lambda+2\mu)w_{1,11}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{3,13}+\frac{k\mu}{k+\mu}w_{1,33}\\
&\quad-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1133}+w_{3,3111}\right)\\
&\quad+\frac{\rho\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,33tt}+w_{3,31tt}\right)=\rho w_{1,tt}\\
&(\lambda+2\mu)w_{3,33}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{1,13}+\frac{k\mu}{k+\mu}w_{3,11}\\
&\quad-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1113}+w_{3,1111}\right)\\
&\quad+\rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,13tt}+w_{3,11tt}\right)=\rho w_{3,tt}
\end{aligned}
$$

and quasi-static 2D system of equations

$$
\begin{aligned}
&(\lambda+2\mu)w_{1,11}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{3,13}+\frac{k\mu}{k+\mu}w_{1,33}\\
&\quad-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1133}+w_{3,3111}\right)=0\\
&(\lambda+2\mu)w_{3,33}+\left(\lambda+\frac{k\mu}{k+\mu}\right)w_{1,13}+\frac{k\mu}{k+\mu}w_{3,11}\\
&\quad-\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1113}+w_{3,1111}\right)=0
\end{aligned}
$$

Finally 1D dynamic or quasi-static system of equations for bending of layered massive (case $w_1=0, w_3=w_3(x_1,t)$) takes the view

$$
\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}w_{3,1111}-\frac{k\mu}{k+\mu}w_{3,11}-\rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}w_{3,11tt}+\rho w_{3,tt}=0
$$

for dynamics, and

$$
\frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}w_{3,1111}-\frac{k\mu}{k+\mu}w_{3,11}=0
$$

for quasi-statics. Formulas for stress tensor components are

$$
\begin{aligned}
\sigma_{ij}^{(0)}&=C_{ijkl}w_{k,l}+C_{ijk3}u_{k,\xi}^{(1)}\\
\sigma_{ij}^{(0)}&=\lambda\delta_{ij}w_{k,k}+\mu(w_{i,j}+w_{j.i})+\mu(\phi_i\delta_{j3}+\phi_j\delta_{i3})
\end{aligned}
$$

$$\sigma_{ij}^{(1)} = C_{ijkl}u_{k,l}^{(1)} + C_{ijk3}u_{k,\xi}^{(2)}$$
$$\sigma_{ij}^{(1)} = \left(\lambda\delta_{ij}\phi_{k,k} + \mu(\phi_{i,j} + \phi_{j.i}) - \lambda\delta_{ij}\psi_3 - \mu(\psi_i\delta_{j3} + \psi_j\delta_{i3})\right)(\xi - sign\xi/2)$$

where $\phi_3 = 0$, $\phi_\gamma = -\mu(w_{\gamma,3} + w_{3,\gamma})/(k+\mu)$, $\psi_\gamma = \phi_{\gamma,3}$, $\psi_3 = \lambda\phi_{\beta,\beta}/(\lambda + 2\mu)$.

Boundary conditions for loaded surface are

$$\sigma_{ij}^{(0)} \cdot n_j = P_i, \;\; \sigma_{ij}^{(1)} \cdot n_j = 0$$

In some problems for definite orientations of boundary normal vector the boundary condition of first order converts into identity. In such cases the boundary condition of second order should be used: $\sigma_{ij}^{(2)} \cdot n_j = 0$.

4 Wave Properties of Layered Medium with Slippage at Inter-layer Boundaries

Below the propagation of plane harmonic and surface Rayleigh waves in layered media is considered.

4.1 Plane Harmonic Waves

Let's define the properties of harmonic waves propagating in arbitrary direction regarding layer orientation at arbitrary inter-layer connection coefficient k. 2D dynamic system of equations for the medium under consideration is

$$\begin{aligned}
&(\lambda + 2\mu)w_{1,11} + \left(\lambda + \frac{k\mu}{k+\mu}\right)w_{3,13} + \frac{k\mu}{k+\mu}w_{1,33} \\
&\quad - \frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1133} + w_{3,3111}\right) \\
&\quad + \rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,33tt} + w_{3,31tt}\right) = \rho w_{1,tt} \\
&(\lambda + 2\mu)w_{3,33} + \left(\lambda + \frac{k\mu}{k+\mu}\right)w_{1,13} + \frac{k\mu}{k+\mu}w_{3,11} \\
&\quad - \frac{\varepsilon^2\mu^3}{3(k+\mu)^2}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\left(w_{1,1113} + w_{3,1111}\right) \\
&\quad + \rho\frac{\varepsilon^2\mu^2}{12(k+\mu)^2}\left(w_{1,13tt} + w_{3,11tt}\right) = \rho w_{3,tt}
\end{aligned}$$

These equations may be rewritten as

$$(\lambda + 2\mu)w_{1,11} + \lambda w_{3,13} + \tilde{\mu}(w_{1,3} + w_{3,1})_{,3} - \varepsilon^2\mu\beta_1 \left(w_{1,3} + w_{3,1}\right)_{,113}$$
$$+ \rho\varepsilon^2\beta_2 \left(w_{1,3} + w_{3,1}\right)_{,3tt} = \rho w_{1,tt}$$
$$(\lambda + 2\mu)w_{3,33} + \lambda w_{1,13} + \tilde{\mu}(w_{1,3} + w_{3,1})_{,1} - \varepsilon^2\mu\beta_1 \left(w_{1,3} + w_{3,1}\right)_{,111}$$
$$+ \rho\varepsilon^2\beta_2 \left(w_{1,3} + w_{3,1}\right)_{,1tt} = \rho w_{3,tt}$$

Introduce the additional variables

$$U = w_{1,3} + w_{3,1}$$
$$V = \tilde{\mu}U - \varepsilon^2\mu\beta_1 U_{,11} + \rho\varepsilon^2\beta_2 U_{,tt}$$

The system of equations takes the following view

$$\left((\lambda + 2\mu)w_{1,11} - \rho w_{1,tt}\right) + \lambda w_{3,13} + V_{,3} = 0$$
$$\lambda w_{1,13} + \left((\lambda + 2\mu)w_{3,33} - \rho w_{3,tt}\right) + V_{,1} = 0$$
$$w_{1,3} + w_{3,1} - U = 0$$
$$\tilde{\mu}u - \varepsilon^2\beta_2(\mu_* u_{,11} + \rho u_{,tt}) - V = 0$$

Here the following designations are introduced

$$\tilde{\mu} = \mu\frac{k}{k+\mu},\ \beta = \frac{\mu}{k+\mu},\ \beta_1 = \frac{\lambda+\mu}{\lambda+2\mu}\beta^2/3,\ \beta_2 = \beta^2/12,\ \mu_* = \mu\beta_1/\beta_2$$

We seek the solution of this system of equations as harmonic waves propagating in the direction $n = (n_1, n_3)$ with frequency ω and wave number $\kappa = \kappa n = (\kappa_1, \kappa_3)$

$$w_1 = Ae^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)},\ w_3 = Be^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)}$$
$$U = Ce^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)},\ V = De^{i(\kappa_1 x_1 + \kappa_3 x_3 - \omega t)}$$

where $\kappa_1 = \kappa n_1$, $\kappa_3 = \kappa n_3$, $|\kappa| = \kappa$, $|n| = 1$, $k = 2\pi/l$ is the wave number, l is harmonic wave length, $\varepsilon k = 2\pi\varepsilon/l$, $\varepsilon^2 k^2 = 4\pi^2(\varepsilon/l)^2$. The value $\varepsilon/l \ll 1$ is a small parameter. In result we get the system of homogeneous algebraic equations

$$\left((\lambda + 2\mu)\kappa_1^2 + \mu_\varepsilon\kappa_3^2 - \rho\omega^2\right) A + (\lambda + \mu_\varepsilon)\kappa_1\kappa_3 B = 0$$
$$(\lambda + \mu_\varepsilon)\kappa_1\kappa_3 A + \left((\lambda + 2\mu)\kappa_3^2 + \mu_\varepsilon\kappa_1^2 - \rho\omega^2\right) B = 0$$

Here $\mu_\varepsilon = \tilde{\mu} + \varepsilon^2\beta_2(\mu_*\kappa_1^2 - \rho\omega^2)$. Condition of the solvability for this algebraic system gives the equation for propagation velocities of harmonic waves in the medium under consideration:

$$\zeta^4 - \left(1 + \frac{\mu_\varepsilon}{(\lambda+2\mu)}\right)\zeta^2 + \frac{\mu_\varepsilon}{(\lambda+2\mu)} + 4\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{(\mu-\mu_\varepsilon)}{(\lambda+2\mu)}n_1^2 n_3^2 = 0$$

Here $\zeta^2 = \rho c^2/(\lambda + 2\mu) = c^2/c_1^2$, $c = \omega/\kappa$ is the phase velocity of wave propagation in layered medium, $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are velocities of elastic longitudinal and transverse waves in a homogeneous elastic medium.

Let α ($n_1 = \sin\alpha$) is the angle of wave propagation direction. For some values of α the biquadratic equation has exact solution.

At $\alpha = 0$ have $\zeta_1 = 1$ and $\zeta_2 = \sqrt{\tilde{\mu}}\Big/\sqrt{(\lambda + 2\mu)(1 + \varepsilon^2\kappa^2\beta_2)}$ for quasi-longitudinal wave and for quasi-transversal wave respectively.

At $\alpha = \pi/4$ have $\zeta_1 = \sqrt{(\lambda + \mu + \tilde{\mu} + \varepsilon^2\kappa^2\beta_2\mu_*/2)}\Big/\sqrt{(\lambda + 2\mu)(1 + \varepsilon^2\kappa^2\beta_2)}$ and $\zeta_2 = \sqrt{\mu}\Big/\sqrt{(\lambda + 2\mu)}$ for quasi-longitudinal wave and for quasi-transversal wave respectively.

At $\alpha = \pi/2$ have $\zeta_1 = 1$ and $\zeta_2 = \sqrt{(\tilde{\mu} + \varepsilon^2\kappa^2\beta_2\mu_*)}/\Big/\sqrt{(\lambda + 2\mu)(1 + \varepsilon^2\kappa^2\beta_2)}$ for quasi-longitudinal wave and for quasi-transversal wave respectively.

At arbitrary α the solution of this equation may be sought in assumed approximation $\sim \varepsilon^2$ as $\zeta^2 = \zeta_0^2 + \zeta_*^2\varepsilon^2 + o(\varepsilon^2)$.

Zero approximation $\zeta = \zeta_0^2$ is found from equation:

$$\zeta_0^4 - \left(1 + \frac{\tilde{\mu}}{(\lambda + 2\mu)}\right)\zeta_0^2 + \frac{\tilde{\mu}}{(\lambda + 2\mu)} + \frac{(\lambda + \mu)}{(\lambda + 2\mu)}\frac{(\mu - \tilde{\mu})}{(\lambda + 2\mu)}\sin^2 2\alpha = 0$$

Values ζ_0^2 which correspond to quasi-longitudinal and quasi-transversal waves in layered medium are:

$$\zeta_0^2 = 0.5\,(1 + \tilde{\mu}/(\lambda + 2\mu) \pm D_0)$$

where

$$D_0 = \sqrt{\frac{(\lambda + \mu)^2}{(\lambda + 2\mu)^2} + 2\frac{(\lambda + \mu)}{(\lambda + 2\mu)}\frac{(\mu - \tilde{\mu})}{(\lambda + 2\mu)}\cos 4\alpha + \frac{(\mu - \tilde{\mu})^2}{(\lambda + 2\mu)^2}}$$

The correction coefficient ζ_*^2 is:

$$\zeta_*^2 = \beta_2\kappa^2(\zeta_0^2 - \cos^2 2\alpha)\left(\frac{\mu_*}{(\lambda + 2\mu)}\sin^2\alpha - \zeta_0^2\right)\left(2\zeta_0^2 - \left(1 + \frac{\tilde{\mu}}{(\lambda + 2\mu)}\right)\right)^{-1}$$

Approximate values of phase velocities with accuracy ε^2 are

$$\zeta \approx \zeta_0\left(1 + \kappa^2\varepsilon^2\beta_2(\zeta_0^2 - \cos^2 2\alpha)\left(\zeta_0^2 - \frac{\mu_*}{(\lambda + 2\mu)}\sin^2\alpha\right)/2\zeta_0^2 D_0\right)$$

From these formulas it is seen that the velocities of harmonic waves have small dispersion ($\sim \kappa^2\varepsilon^2$) and depend on the wave direction parameter α.

Now investigate the limit cases of these formulas at $\varepsilon \to 0$ ($\mu_\varepsilon \to \tilde{\mu}$). Firstly it is the limit case of ideal inter-layer contact (case of homogeneous elastic medium): $k \to \infty$ ($\tilde{\mu} \to \mu$), and secondly it is the limit case of ideal inter-layer slipping $k \to 0$ ($\tilde{\mu} \to 0$).

Quasi-longitudinal waves (sign plus in formulas for ζ_0 andζ).

In this case for $\varepsilon \to 0$: $\zeta \to \zeta_0$.

For $k \to \infty$: $\zeta_0 \to 1$ ($c \to c_1$), (elastic longitudinal wave in isotropic medium).

For $k \to 0 : \zeta_0^2 \to 0.5\,(1 + D_1)$

Here

$$D_1 = \sqrt{\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2} + \frac{2(\lambda+\mu)\mu}{(\lambda+2\mu)^2}\cos 4\alpha + \frac{\mu^2}{(\lambda+2\mu)^2}}$$

For $\alpha = 0, \pi/2$: $\zeta_0 \to 1, c \to c_1$, (waves along and cross layers).

For $\alpha = \pi/4$:$\zeta_0 \to \sqrt{(\lambda+\mu)/(\lambda+2\mu)}$ (waves propagated under an angle to the layer boundary direction, minimal propagation velocity).

Quasi-transversal waves (sign minus in formulas for ζ_0 and ζ).

In this case for $\varepsilon \to 0$: $\zeta \to \zeta_0$.

For $k \to \infty$: $\zeta \to c_2/c_1$ ($c \to c_2$), (elastic transversal wave in isotropic medium).

For $k \to 0$: $\zeta_0^2 \to 0.5\,(1 - D_1)$.

For $\alpha = 0, \pi/2$: $\zeta_0 \to 0$, $c \to 0$, (waves along and cross layers).

For $\alpha = \pi/4 : \zeta_0 \to c_2/c_1$, $c \to c_2$, (waves propagated under an angle to the layer boundary direction, maximal propagation velocity).

The dependence of propagation velocities for quasi-longitudinal and quasi-transversal waves on coefficients of inter-layer connection k are shown in Fig. 1. Upper graphs correspond to quasi-longitudinal waves, lower graphs correspond to quasi-transversal waves at various values of small parameter ε/l = 0.5, 0.3, 0.1. Dimensionless elastic moduli are defined as $\lambda/(\lambda+2\mu) = \mu/(\lambda+2\mu) = 1/3$.

Above each graph the value of wave direction angle $= 0, 30°, 60°, 90°$ is shown. For $= 0, 90°$ the solutions are described by exact formulas given above and shown in Fig. 1a, d. For other values of the solution of biquadratic equation for $\zeta = c/c_1$ is calculated numerically and shown in Fig. 1b, c.

From these graphs the level of plane wave dispersion can be seen (for small values of the coefficient of inter-layer connection) for various wave directions. The dependence of dispersion on the layer thickness parameter ε/l can also be seen there. It is possible to conclude that the dispersion plays role only for dimensionless coefficients of inter-layer connection $k/(\lambda + 2\mu) < 0.7$. It is mostly significant for directions $= 90°$ (along layers) of quasi-transversal waves (see Fig. 1d, lower graphs).

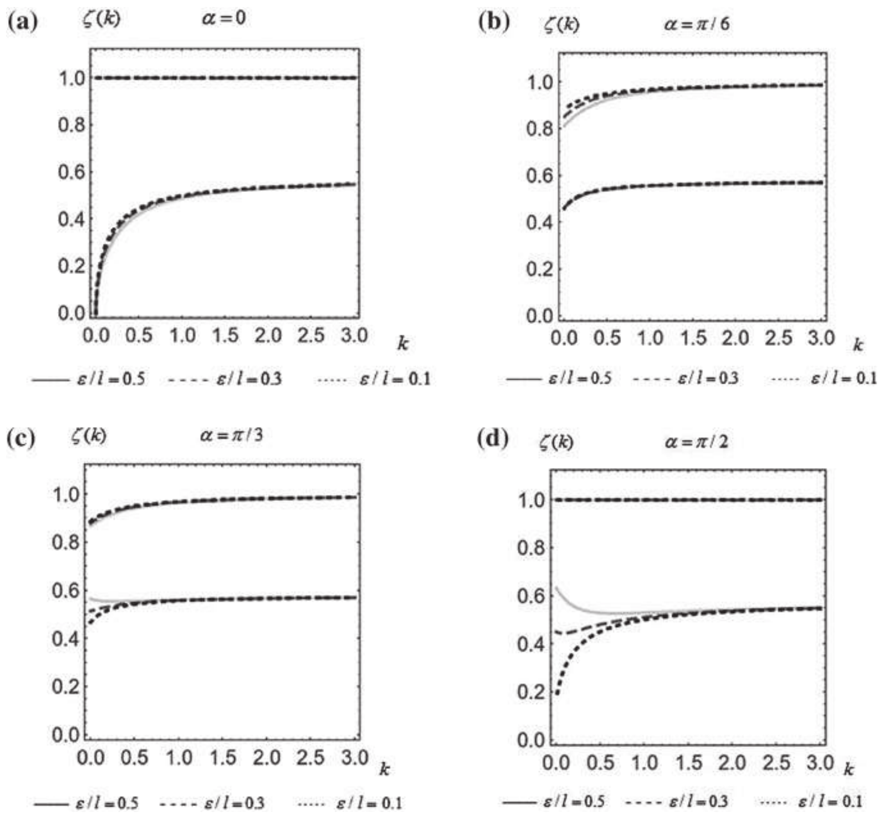

Fig. 1 The dependence of dimensionless velocities for quasi-longitudinal and quasi-transversal waves on coefficients of inter-layer connection k

4.2 Surface Rayleigh Waves

Consider surface waves on the boundary of layered half-plane $-\infty < x_3 \leq 0$, $-\infty < x_1 < \infty$ (plane task). The system of equations for displacements of layered medium with slippage at inter-layer boundaries is written earlier

$$
\begin{aligned}
&\big((\lambda + 2\mu)w_{1,11} - \rho w_{1,tt}\big) + \lambda w_{3,13} + V_{,3} = 0 \\
&\lambda w_{1,13} + \big((\lambda + 2\mu)w_{3,33} - \rho w_{3,tt}\big) + V_{,1} = 0 \\
&w_{1,3} + w_{3,1} - U = 0, \tilde{\mu}U - \varepsilon^2\beta_2(\mu_* U_{,11} + \rho U_{,tt}) - V = 0
\end{aligned}
$$

Boundary conditions at $x_3 = 0$

$$\sigma_{33} = (\lambda + 2\mu)w_{3,3} + \lambda w_{1,1} = 0, \sigma_{13} = \mu(w_{1,3} + w_{3,1}) = 0$$

At $x_3 \to -\infty$ have $w_1 \to 0$, $w_3 \to 0$.

Represent the solutions of this task as surface wave ($\gamma > 0$)

$$w_1 = Ae^{\gamma x_3} e^{i(\kappa_1 x_1 - \omega t)}, \; w_3 = Be^{\gamma x_3} e^{i(\kappa_1 x_1 - \omega t)}$$

Substituting this representation in to the system of differential equations we get the algebraic homogeneous system of equations

$$\begin{gathered}\left(\mu_\varepsilon \gamma^2 - \kappa_1^2 \Delta_1\right) A + (\lambda + \mu_\varepsilon)\gamma i \kappa_1 B = 0 \\ -\kappa_1^2 (\lambda + \mu_\varepsilon)\gamma A + \left((\lambda + 2\mu)\gamma^2 - \kappa_1^2 \Delta_{2\varepsilon}\right) i\kappa_1 B = 0\end{gathered}$$

Here the following designations are used: $\mu_\varepsilon = \tilde{\mu} + \varepsilon^2 \beta_2 \kappa_1^2 \Delta_*$, $\Delta_* = \mu_* - \rho c^2$, $\Delta_1 = \lambda + 2\mu - \rho c^2$, $\Delta_{2\varepsilon} = \Delta_2 + \varepsilon^2 \beta_2 \kappa_1^2 \Delta_*$, $\Delta_2 = \tilde{\mu} - \rho c^2$.

Phase velocity of surface wave is $c = \omega/\kappa_1$. The solvability condition gives the biquadratic equation for γ

$$(\lambda + 2\mu)\mu_\varepsilon \gamma^4 - \kappa_1^2 \gamma^2 D_2 + \kappa_1^4 \Delta_1 \Delta_{2\varepsilon} = 0$$

where $D_2 = \mu_\varepsilon \Delta_{2\varepsilon} + (\lambda + 2\mu)\Delta_1 - (\lambda + \mu_\varepsilon)^2$.

From this equation we find two positive solutions$\gamma_{1,2} > 0$

$$\gamma_{1,2}^2 = \frac{\kappa_1^2 \left\{ D_2 \pm \sqrt{D_2^2 - 4(\lambda + 2\mu)\mu_\varepsilon \Delta_1 \Delta_{2\varepsilon}} \right\}}{2(\lambda + 2\mu)\mu_\varepsilon}$$

Then the solutions of task are

$$\begin{aligned} w_1 &= A_1 e^{\gamma_1 x_3} e^{i(\kappa_1 x_1 - \omega t)} + A_2 e^{\gamma_2 x_3} e^{i(\kappa_1 x_1 - \omega t)} \\ w_3 &= B_1 e^{\gamma_1 x_3} e^{i(\kappa_1 x_1 - \omega t)} + B_2 e^{\gamma_2 x_3} e^{i(\kappa_1 x_1 - \omega t)} \end{aligned}$$

where $i\kappa_1 B_{1,2} = \kappa_1^2 (\lambda + \mu_\varepsilon)\gamma_{1,2} A_{1,2} \left((\lambda + 2\mu)\gamma_{1,2}^2 - \kappa_1^2 \Delta_{2\varepsilon}\right)^{-1}$

Substituting these solutions into boundary conditions at $x_3 = 0$ get the system of equations

$$\begin{gathered}\gamma_1 A_1 + \gamma_2 A_2 + i\kappa_1 B_1 + i\kappa_1 B_2 = 0 \\ -\lambda \kappa_1^2 A_1 - \lambda \kappa_1^2 A_2 + (\lambda + 2\mu)\gamma_1 i\kappa_1 B_1 + (\lambda + 2\mu)\gamma_2 i\kappa_1 B_2 = 0\end{gathered}$$

From this system of equations the amplitudes B_1 and B_2 may be excluded. Then we have two homogeneous equations regarding amplitudes A_1 and A_2. For simplification of expressions instead of $\gamma_{1,2} > 0$ introduce values $\eta_{1,2}$ from relations $\eta_{1,2} = \gamma_{1,2}/\kappa_1$. These values are defined by formulas

$$\eta_{1,2}^2 = \frac{D_2 \pm \sqrt{D_2^2 - 4(\lambda + 2\mu)\mu_\varepsilon \Delta_1 \Delta_{2\varepsilon}}}{2(\lambda + 2\mu)\mu_\varepsilon}$$

Homogeneous system of equations for amplitudes A_1 and A_2 is

$$\eta_1\left(1+\frac{(\lambda+\mu_\varepsilon)}{\left((\lambda+2\mu)\eta_1^2-\Delta_{2\varepsilon}\right)}\right)A_1+\eta_2\left(1+\frac{(\lambda+\mu_\varepsilon)}{\left((\lambda+2\mu)\eta_2^2-\Delta_{2\varepsilon}\right)}\right)A_2=0$$
$$\left(\frac{(\lambda+2\mu)(\lambda+\mu_\varepsilon)\eta_1^2}{\left((\lambda+2\mu)\eta_1^2-\Delta_{2\varepsilon}\right)}-\lambda\right)A_1+\left(\frac{(\lambda+2\mu)(\lambda+\mu_\varepsilon)\eta_2^2}{\left((\lambda+2\mu)\eta_2^2-\Delta_{2\varepsilon}\right)}-\lambda\right)A_2=0$$

For solvability the determinant of this system should be equal to zero. It gives the equation for unknown phase velocity of surface wave $c=\omega/\kappa_1$

$$\begin{aligned}&4(\lambda+\mu)\eta_1\eta_2^2-\eta_2(1+\eta_2^2)\left((\lambda+2\mu)\eta_1^2+\lambda\eta_2^2\right)\\&\quad-\frac{\Delta\mu_\varepsilon}{\mu}\left\{\eta_1\left((\lambda+2\mu)\eta_2^2+\lambda\right)+\eta_2(1+\eta_2^2)\left((\lambda+2\mu)\eta_1^2+\lambda\right)\right\}=0\end{aligned}$$

Here we denote $\Delta\mu_\varepsilon=\mu-\mu_\varepsilon$. Again investigate the limit cases of this formula at $\varepsilon\to 0$ ($\mu_\varepsilon\to\tilde{\mu}$). In these cases

$$\eta_{1,2}^2=\frac{\tilde{D}_3\pm\sqrt{\tilde{D}_3^2-4(\lambda+2\mu)\tilde{\mu}\Delta_1\Delta_2}}{2(\lambda+2\mu)\tilde{\mu}}$$

where $\tilde{D}_3=\tilde{\mu}\Delta_2+(\lambda+2\mu)\Delta_1-(\lambda+\tilde{\mu})^2$.

The equation for surface wave propagation velocity is

$$\begin{aligned}&4(\lambda+\mu)\eta_1\eta_2^2-\eta_2(1+\eta_2^2)\left((\lambda+2\mu)\eta_1^2+\lambda\eta_2^2\right)\\&\quad-\frac{\mu}{(k+\mu)}\left\{\eta_1\left((\lambda+2\mu)\eta_2^2+\lambda\right)+\eta_2(1+\eta_2^2)\left((\lambda+2\mu)\eta_1^2+\lambda\right)\right\}=0\end{aligned}$$

Case of ideal contact (ideal elastic medium)

In this case at $k\to\infty$ *(*$\tilde{\mu}\to\mu$*)*:

$$\eta_1^2=1-c^2/c_1^2,\ \eta_2^2=1-c^2/c_2^2,\ \ 4(\lambda+\mu)\eta_1\eta_2-(1+\eta_2^2)\left((\lambda+2\mu)\eta_1^2+\lambda\eta_2^2\right)=0$$

After short transformation we come to classic Rayleigh wave:

$$4\sqrt{1-c^2/c_1^2}\sqrt{1-c^2/c_2^2}-(2-c^2/c_2^2)^2=0$$

Case of ideal inter-layer slipping

In this case at $k\to 0$ *(*$\tilde{\mu}\to 0$*)* treating μ_ε as small parameter we get:

$$\eta_1^2\sim\frac{4\mu(\lambda+\mu)-(\lambda+2\mu)\rho c^2}{(\lambda+2\mu)\mu_\varepsilon},\ \eta_2^2\sim\frac{(\lambda+2\mu-\rho c^2)(\mu_\varepsilon-\rho c^2)}{4\mu(\lambda+\mu)-(\lambda+2\mu)\rho c^2}$$
$$(3\lambda+2\mu)\eta_1\eta_2^2-2(\lambda+2\mu)\eta_1^2\eta_2(1+\eta_2^2)-\lambda\eta_2(1+\eta_2^2)^2-\lambda\eta_1=0$$

The graphs for dependence of dimensionless surface wave velocity c/c_1 on inter-layer connection coefficient k is shown in Fig. 2 for various values of layer thickness parameter $\varepsilon/l = 0.5, 0.3, 0.1$. As in previous case the wave number is $\kappa_1 = 2\pi/l$, where l is the length of harmonic surface wave. The asymptotic of classic Rayleigh root takes place for $k/(\lambda + 2\mu) > 1.5 \div 2$. These graphs are very similar to the lower graphs in Fig. 1d (quasi-transversal waves) for waves propagating along layers ($= 90°$) and very close to them. For classic Rayleigh waves, as it is known, $c_R/c_2 \approx 0.9$, the same relation is valid and in the case under consideration for ratio of velocity of surface waves to the velocity of quasi-transversal waves.

Remark that the applicability boundary of proposed asymptotic theory is not defined exactly. The upper boundary for small parameter $\varepsilon/l = 0.5$ is assumed quite approximately. Nevertheless, for inter-layer connection coefficients starting from values $k/(\lambda + 2\mu) > 0.7$, the calculations give very close meanings for propagation velocity of quasi-longitudinal, quasi-transversal and surface waves for the whole range of wave lengths $\varepsilon/l < 0.5$.

It should be noted that proposed refined theory may be used for investigation of transformation seismic waves exiting to the day surface in rock massifs with regular parallel crack grids accounting slippage at contact boundaries. Also this theory may be useful for description of composite materials with additional soft sublayers between more rigid layers.

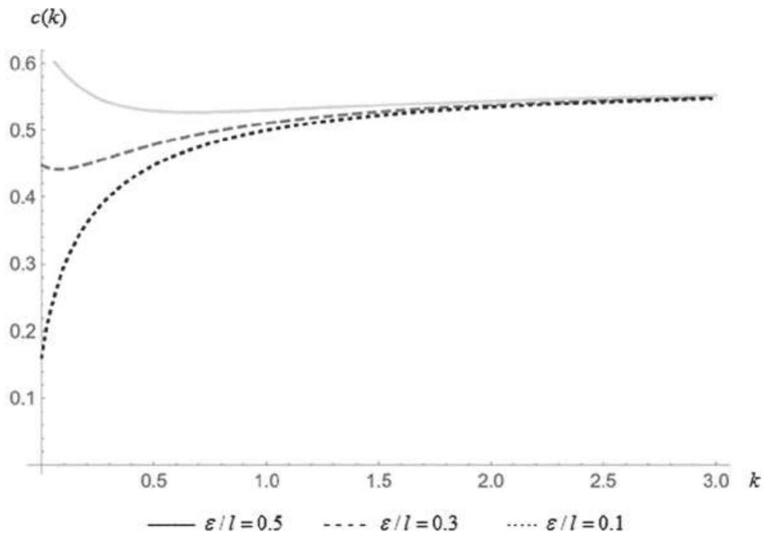

Fig. 2 The dependence of dimensionless surface wave velocity on inter-layer connection coefficient k

5 Conclusion

Using the asymptotic method of homogenisation the continuum theory of layered medium is built taking into account terms of second order accuracy regarding the small parameter of layer thickness. The linear slip contact condition is used to describe the relation between tangential displacement jumps and shear stresses. The wave properties of the proposed refined equations are studied, the dispersion relations are derived and the propagation of harmonic waves is investigated. The problem of surface Rayleigh like waves is solved.

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