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Continuum Model of the Layered Medium with Slippage and Nonlinear Conditions at the Interlayer Boundaries

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Abstract. The equations for layered medium with slippage are obtained using the asymptotic method of homogenization. The terms of second order respectively the small parameter of layer thickness are taken into account. The linear slip condition defines the dependence between the tangential jumps of displacements at the contact boundary and the shear stresses. Such generalized models are needed in the study of static and dynamic deformations of layered rock media. Also these models may be useful for description of composite materials with additional soft sublayers between more rigid layers.

Introduction

In our study the homogenized models of layered medium with slippage are derived by using asymptotic method [1,2]. The second order terms relatively small parameters of layer thickness are taken in to account. The linear and nonlinear slip relations between tangential displacement jumps at interlayer boundaries and shear stresses are used. The zero order approximate equations for such media has been derived earlier in [3,4]. Such generalized models are required for static and dynamic problems of rock media deformations and for dynamic wave propagation problems in geophysics. Often these rocks contain regular grid of cracks which can be considered as layered structures. Classical studies of wave fields in such media usually are based on assumption of continuity of displacement fields. But for rather strong seismic actions the possibility of tangential displacement jumps at the interlayer boundaries should be taken into account. It should be noted also that the theory of layered media is suitable for description of composite materials with soft (rubber) sublayers between major more rigid (metallic) layers.

1. Improved model for linear Winkler type conditions at interlayer boundaries. We consider infinite layered medium using Cartesian rectangular coordinate system (x_1, x_2, x_3). The axis x_3 is perpendicular to the planes of parallel flat boundaries between layers. Let the interlayer boundaries have coordinates $x_3 = x^{(s)} = s\varepsilon$ (s=0, $\pm1, \pm2, \ldots$), where constant layer thickness $\varepsilon / l << 1$ is a small parameter, where l is the size of distributed load application range, for instance, wave length in the processes under consideration. In such case all spatial values should be made dimensionless using this value l.

Assume that layer boundaries are always compressed and the following conditions are valid:

$\sigma_{33} < 0$, $[u_3] = [\sigma_{\gamma 3}] = [\sigma_{33}] = 0$, $\sigma_{\gamma 3} = k_* [u_\gamma]$

It is linear slippage of Winkler type, $k_* \varepsilon = k = O(1)$. Square brackets $[f] = f|_{x^{(s)}+0} - f|_{x^{(s)}-0}$ designate the jump of a value f at inter-layer boundary. Such conditions are valid approximately if between layers the soft sublayers of thickness δ ($\delta / \varepsilon << 1$) with small shear modulus μ_δ are present. In this case $\mu_\delta = k\delta / \varepsilon$ or vice versa $k = \mu_\delta \varepsilon / \delta$. It is possible to say that k is inter-layer shear connection coefficient. The layers themselves are elastic isotropic and subjected to momentum equations and Hooke's law:

$x_3 \neq x^{(s)}$: $\sigma_{ij,j} - \rho u_{i,tt} = 0$, $\sigma_{ij} = C_{ijkl} u_{k,l}$

Here the elastic moduli tensor is:

$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$

According to the method of asymptotic homogenization [1] let's introduce «fast» variable $\xi = x_3 / \varepsilon$. According to [1] assume that $u_k = u_k(x_l, \xi, t)$ is a function, which is smooth regarding «slow» variables x_l and continuous regarding «fast» variable ξ, excluding points $\xi^{(s)} = x^{(s)} / \varepsilon$, where it may have jumps of first kind. Besides, along ξ the displacement is 1-periodic $[[u_i]] = u_i\big|_{\xi^{(s)}+1/2} - u_i\big|_{\xi^{(s)}-1/2} = 0$. Accounting such choice of variables and the differentiation rule for complex functions, the system of equations for cell of periodicity $x^{(s)} - 1/2 \leq x_3 \leq x^{(s)} + 1/2$, $-1/2 \leq \xi \leq 1/2$ can be rewritten as

$x_3 \neq x^{(s)}$, $\xi \neq 0$: $\varepsilon^{-2} C_{i3k3} u_{k,\xi\xi} + \varepsilon^{-1}(C_{ijk3} u_{k,j\xi} + C_{i3kl} u_{k,l\xi}) + C_{ijkl} u_{k,lj} - \rho u_{i,tt} = 0$

The contact conditions are

$x_3 = x^{(s)}$, $\xi = 0$: $\varepsilon^{-1} C_{33k3} u_{k,\xi} + C_{33kl} u_{k,l} < 0$

$[u_3]=0$, $[\varepsilon^{-1} C_{i3k3} u_{k,\xi} + C_{i3kl} u_{k,l}] = 0$, $\varepsilon^{-1} C_{\gamma 3k3} u_{k,\xi} + C_{\gamma 3kl} u_{k,l} = k_* [u_\gamma]$

The conditions of 1-periodicity are

$[[u_i]] = u_i\big|_{\xi+1/2} - u_i\big|_{\xi-1/2} = 0$.

Here and farther, Greek indices (β, γ) take values 1 and 2, Latin indices take values 1, 2, 3. The displacements are represented as asymptotic series regarding small parameter ε:

$u_i = u_i^{(0)}(x_k, \xi, t) + \varepsilon u_i^{(1)}(x_k, \xi, t) + \varepsilon^2 u_i^{(2)}(x_k, \xi, t) + \varepsilon^3 u_i^{(3)}(x_k, \xi, t) + \ldots$

Introduce the operation of «averaging» $\langle f \rangle$ for the function of «fast» variable ξ, which will be often used farther: $\langle f \rangle = \int_{-1/2}^{1/2} f d\xi$. Displacement approximations should satisfy the additional condition $\langle u_k^{(n)} \rangle = 0$ [1].

Substitute this representation into the theory of elasticity equations. Equating to zero the term with negative power ε^{-2} we get that zero approximation $u_i^{(0)}$ is independent on the «fast» variable ξ and $u_i^{(0)} = w_i(x_k, t)$. Equating to zero the term with negative power ε^{-1} we get that first approximation $u_i^{(1)}$ satisfies the equation $C_{i3k3} u_{k,\xi\xi}^{(1)} = 0$. The resulting system of differential equations is:

$$C_{ijkl} w_{k,jl} + C_{ijk3} u_{k,j\xi}^{(1)} + (C_{i3kl} u_{k,l}^{(1)} + C_{i3k3} u_{k,\xi}^{(2)})_{,\xi} +$$
$$+\varepsilon \left[C_{ijkl} u_{k,jl}^{(1)} + C_{ijk3} u_{k,j\xi}^{(2)} + (C_{i3kl} u_{k,l}^{(2)} + C_{i3k3} u_{k,\xi}^{(3)})_{,\xi} \right] +$$
$$+\varepsilon^2 \left[C_{ijkl} u_{k,jl}^{(2)} + C_{ijk3} u_{k,j\xi}^{(3)} + (C_{i3kl} u_{k,l}^{(3)} + C_{i3k3} u_{k,\xi}^{(4)})_{,\xi} \right] + \ldots = \rho w_{i,tt} + \varepsilon \rho u_{i,tt}^{(1)} + \varepsilon^2 \rho u_{i,tt}^{(2)} + \ldots$$

A similar representation for stress tensor components is:

$\sigma_{ij} = \sigma_{ij}^{(0)} + \varepsilon \sigma_{ij}^{(1)} + \varepsilon^2 \sigma_{ij}^{(2)} + \ldots$

where $\sigma_{ij}^{(n)} = C_{ijkl} u_{k,l}^{(n)} + C_{ijk3} u_{k,\xi}^{(n+1)}$.

All approximations for stresses are 1-periodic functions of ξ. In particular, the relation $\sigma_{i3}^{(n)} = C_{i3kl} u_{k,l}^{(n)} + C_{i3k3} u_{k,\xi}^{(n+1)}$ and conditions $[\sigma_{i3}^{(n)}] = 0$, $[[\sigma_{i3}^{(n)}]] = 0$ are valid. It is easy to see that $\langle \sigma_{i3\,,\xi}^{(n)} \rangle = 0$.

Accounting the terms of definite order of ε, applying the averaging operation $\langle f \rangle$ and excluding the dependence on «fast» variable ξ, we get the model of homogenized layered medium with slippage of Winkler type.

Let's derive the improved theory of second order. For this in the system of equations we keep the terms of order ε^2. Applying averaging operation $\langle\ \rangle$ for periodicity cell to the system of equations we get the following:

$$C_{ijkl}w_{k,jl}+C_{ijk3}\left\langle u^{(1)}_{k,\xi}\right\rangle_{,j}+\varepsilon C_{ijk3}\left\langle u^{(2)}_{k,\xi}\right\rangle_{,j}+\varepsilon^2 C_{ijk3}\left\langle u^{(3)}_{k,\xi}\right\rangle_{,j}=\rho w_{i,tt}$$

It is the final homogenized system of equations for layered medium with slippage. For complete formulations it needs to find the functions $\left\langle u^{(n)}_{k,\xi}\right\rangle$ *(n=1,2,3)*, which participate in the system. Every function $u^{(n)}_i(x_k,\xi,t)$ *(n=1,2,3)* is found from the appropriate task on «periodicity cell» $(-1/2\le\xi\le 1/2)$ [1], which is formulated by equating to zero the sum of terms of definite order ε^{n-1} in asymptotic system of equations. Additional conditions for these functions can be received by reformulating the contact inter-layer conditions for each function: conditions of 1-periodicity $[[u^{(n)}_i]]=0$ and conditions $\left\langle u^{(n)}_i\right\rangle=0$.

After solving of the tasks on «periodicity cell» we can formulate the refined system of equations for layered medium with slippage:

$$(\lambda+\mu)w_{k,k\gamma}+\mu w_{\gamma,kk}+\mu\varphi_{\gamma,3}+\varepsilon^2\mu^2\left(\varphi_{\gamma,\beta\beta3}+(3\lambda+2\mu)\varphi_{\beta,\beta\gamma3}/(\lambda+2\mu)-\rho\varphi_{\gamma,tt3}/\mu\right)/(k+\mu)/12=\rho w_{\gamma,tt}$$

$$(\lambda+\mu)w_{k,k3}+\mu w_{3,kk}+\mu\varphi_{\beta,\beta}+\varepsilon^2\mu^2\left(4(\lambda+\mu)\varphi_{\beta,\beta\alpha\alpha}/(\lambda+2\mu)-\rho\varphi_{\beta,\beta tt}/\mu\right)/(k+\mu)/12=\rho w_{3,tt}$$

$$\varphi_\gamma=-\mu(w_{\gamma,3}+w_{3,\gamma})/(k+\mu).$$

In general equations the expressions for φ_γ are not substituted to avoid the unnecessary complexity of formulas. It is seen that regarding spatial variables this is the system of forth order for the displacements w_k and it contains mixed time derivatives.

On the basis of 2D dynamic system the properties of harmonic longitudinal and transversal waves propagation were defined for arbitrary direction regarding layer orientation [5]. Also there the dispersion relations for arbitrary inter-layer connection coefficient k were derived and the problem of surface Rayleigh waves propagation in layered media was investigated in refined settings.

2. Improved model for linear viscous conditions at interlayer boundaries. In this case it is assumed that the layers are always compressed and that the linear viscous slip conditions are satisfied:

$$\sigma_{33}<0\quad [u_3]=[\sigma_{\gamma 3}]=[\sigma_{33}]=0,\ \ \sigma_{\gamma 3}=\eta[v_\gamma]/\varepsilon$$

Using the method and the results obtained above improved system of equations is derived as follows:

$$(\lambda+\mu)w_{k,k\gamma}+\mu w_{\gamma,kk}+\mu\varphi_{\gamma,3}-\varepsilon^2\mu\Omega_{\gamma,3}=\rho w_{\gamma,tt}$$

$$(\lambda+\mu)w_{k,k3}+\mu w_{3,kk}+\mu\varphi_{\beta,\beta}-\varepsilon^2\mu\Omega_{\beta,\beta}=\rho w_{3,tt}$$

$$\varphi_\gamma=-\int_0^t\tau_\gamma e^{-\mu(t-t_1)/\eta}dt_1/\eta,\qquad \tau_\gamma=\mu(w_{\gamma,3}+w_{3,\gamma}),\qquad \psi_\gamma=\varphi_{\gamma,3},$$

$$\chi_\gamma=-\varphi_{\gamma,ll}-(\lambda+\mu)\varphi_{\beta,\beta\gamma}/\mu+2\psi_{\gamma,3}+(\lambda+\mu)\psi_{3,\gamma}/\mu+\rho\varphi_{\gamma,tt}/\mu\ ,$$

$$\Omega_\gamma=\int_0^t\mu g_\gamma e^{-\mu(t-t_1)/\eta}dt_1/\eta,\quad g_\gamma=(\chi_\gamma-\psi_{\gamma,3}-\psi_{3,\gamma})/12$$

This system of equations for displacements $w_i(x_k,t)$, *i,k*=1,2,3 contains additional functions φ_γ and Ω_γ, γ=1,2. These functions represent the tangential velocity jumps at layer boundaries.

3. Improved model for nonlinear visco-plastic conditions at interlayer boundaries.

The visco-plastic properties are taken into account at interlayer boundaries. It is assumed that up to definite limit of shear stress the slippage is absent and it appears otherwise. These conditions can be formulated in various ways.

We choose a relatively simple form of such slippage conditions: $\eta[v_\gamma]/\varepsilon = \sigma_{\gamma 3} H\left(\sigma_{\beta 3}\sigma_{\beta 3}/\sigma_s{}^2 - 1\right)$. Here $H(y)$ - Heaviside function, $H(y)$=0 at $y<0$, $H(y)$=1 at $y \geq 0$. However, the procedure of asymptotic expansions substitution for velocities and stresses is not correct for the highly nonlinear (breaking) slip condition. Therefore, modify this condition by using smoothed Heaviside function:

$$\eta[v_\gamma]/\varepsilon = \sigma_{\gamma 3} H_d\left(\sigma_{\beta 3}\sigma_{\beta 3}/\sigma_s{}^2 - 1\right).$$

Here $H_d(y)$ - smoothed Heaviside function, $H_d(y) \to H(y)$ at $d \to 0$, d is "effective" width of smoothing. To be specific, we assume the following particular expression for $H_d(y)$ (one of the possible):

$$H_d(y) = 1/2 + arctg\left(y/d\right)/\pi\,,\quad H_d{}^{/}(y) = d/\left(\pi(d^2+y^2)\right),\quad H_d{}^{//}(y) = -2dy/\left(\pi(d^2+y^2)^2\right)$$

Finally, a homogenized system of equations is as follows:

$$(\lambda+\mu)w_{k,k\gamma} + \mu w_{\gamma,kk} + \mu\varphi_{\gamma,3} - \varepsilon^2\mu\Omega_{\gamma,3} = \rho w_{\gamma,tt}$$

$$(\lambda+\mu)w_{k,k3} + \mu w_{3,kk} + \mu\varphi_{\beta,\beta} - \varepsilon^2\mu\Omega_{\beta,\beta} = \rho w_{3,tt}$$

$$\varphi_{\gamma,t} = -\sigma_{\gamma 3}^{(0)} H_d(\Delta_0/d)/\eta\,,\quad \sigma_{\gamma 3}^{(0)} = \mu(w_{\gamma,3} + w_{3,\gamma}) + \mu\varphi_\gamma$$

$$\eta\Omega_{\gamma,t} + \mu\Omega_\gamma H_d\left(\Delta_0/d\right) + \sigma_{\gamma 3}^{(0)}\left(2\mu\sigma_{\beta 3}^{(0)}\Omega_\beta/\sigma_s^2\right)d/\left(\pi(d^2+\Delta_0{}^2)\right) =$$

$$= \mu g_\gamma H_d\left(\Delta_0/d\right) + \sigma_{\gamma 3}^{(0)}\left(2\mu\sigma_{\beta 3}^{(0)} g_\beta/\sigma_s^2\right)d/\left(\pi(d^2+\Delta_0{}^2)\right)$$

$$\Delta_0 = \sigma_{\beta 3}^{(0)}\sigma_{\beta 3}^{(0)}/\sigma_s^2 - 1\,,\quad \psi_\gamma = \varphi_{\gamma,3}\,,\quad g_\gamma = (\chi_\gamma - \psi_{\gamma,3} - \psi_{3,\gamma})/12\,,$$

$$\chi_\gamma = -\varphi_{\gamma,ll} - (\lambda+\mu)\varphi_{\beta,\beta\gamma}/\mu + 2\psi_{\gamma,3} + (\lambda+\mu)\psi_{3,\gamma}/\mu + \rho\varphi_{\gamma,tt}/\mu$$

Thus, the derived system of equations for displacements $w_i(x_k,t)$ contains additional functions φ_γ and Ω_γ too. These functions are subjected to special nonlinear differential equations depending on the choice of contact conditions.

Conclusions

Using asymptotic homogenization method the continuum theory of layered medium is built taking into account terms of second order accuracy regarding the small parameter of layer thickness. The various slip contact conditions are used to describe the relation between tangential displacement jumps and shear stresses.

The improved models for linear Winkler type interlayer conditions, for linear viscous and nonlinear visco-plastic interlayer conditions are derived.

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